

LIE ALGEBRAS RESPONSIBLE FOR ZERO-CURVATURE REPRESENTATIONS OF SCALAR EVOLUTION EQUATIONS

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ABSTRACT. Zero-curvature representations (ZCRs) are well known to be one of the main tools in soliton theory. In particular, Lax pairs in the $(1+1)$ -dimensional case can be interpreted as ZCRs.

For any $(1+1)$ -dimensional scalar evolution equation, we define a sequence of Lie algebras \mathbb{F}^n , $n = 0, 1, 2, 3, \dots$, which classify all ZCRs of this equation up to gauge transformations. The algebra \mathbb{F}^n classifies ZCRs whose x -part depends on jets of order not greater than n .

We prove some results on generators of \mathbb{F}^n . This allows us to compute the explicit structure of \mathbb{F}^n for some examples. In particular, we study the structure of \mathbb{F}^n for equations of the form $u_t = u_{2q+1} + f(x, t, u, u_1, \dots, u_{2q-1})$ for all $q > 0$, which include KdV, Kaup-Kupershmidt, Sawada-Kotera type equations. Here u_k is the k -th derivative of $u = u(x, t)$ with respect to x . For such equations, it is shown that the algebra \mathbb{F}^n is isomorphic to a central extension of the algebra \mathbb{F}^{n-1} for all $n > 2q - 2$. This result allows one to obtain necessary conditions for integrability of such equations.

Some applications to classification of scalar evolution equations with respect to Bäcklund transformations are also discussed.

1. INTRODUCTION AND THE MAIN RESULTS

1.1. Zero-curvature representations and the algebras $\mathbb{F}^n(\mathcal{E}, a)$. We study $(1+1)$ -dimensional scalar evolution equations

$$(1) \quad \frac{\partial u}{\partial t} = F(x, t, u_0, u_1, \dots, u_d), \quad u = u(x, t), \quad u_k = \frac{\partial^k u}{\partial x^k}, \quad u_0 = u.$$

This class of PDEs includes many celebrated equations of mathematical physics (e.g., the KdV, Burgers, Krichever-Novikov, Kaup-Kupershmidt, Sawada-Kotera equations).

In this paper, integrability of PDEs is understood in the sense of soliton theory and the inverse scattering method. This is sometimes called S -integrability.

It is well known that, in order to understand possible integrability properties of (1), one needs to study so-called zero-curvature representations. A *zero-curvature representation* (ZCR) with values in a Lie algebra \mathfrak{g} is given by \mathfrak{g} -valued functions

$$(2) \quad A = A(x, t, u_0, u_1, \dots, u_n), \quad B = B(x, t, u_0, u_1, \dots, u_{n+d-1})$$

satisfying

$$(3) \quad D_x(B) - D_t(A) + [A, B] = 0.$$

The *total derivative operators* D_x, D_t in (3) are given by the formulas

$$(4) \quad D_x = \frac{\partial}{\partial x} + \sum_{k \geq 0} u_{k+1} \frac{\partial}{\partial u_k}, \quad D_t = \frac{\partial}{\partial t} + \sum_{k \geq 0} D_x^k(F(x, t, u_0, u_1, \dots, u_d)) \frac{\partial}{\partial u_k}.$$

We assume that \mathfrak{g} is finite-dimensional and all considered functions are analytic.

The number n in (2) is such that the function A may depend only on the variables x, t, u_k for $k \leq n$. Then equation (3) implies that the function B may depend only on $x, t, u_{k'}$ for $k' \leq n+d-1$, where d is the order of the right-hand side of (1).

Such ZCRs are said to be of *order* $\leq n$. In other words, a ZCR given by A, B is of order $\leq n$ iff $\frac{\partial A}{\partial u_l} = 0$ for all $l > n$.

In this paper, we study the following problem. How to describe all ZCRs (2), (3) for a given equation (1)?

In the case when $n = 0$ and the functions F, A, B do not depend on x, t , a partial answer to this question is provided by the Wahlquist-Estabrook prolongation method (WE method for short). Namely, for a given equation (1), the WE method constructs a Lie algebra in terms of generators and relations such that ZCRs of the form

$$(5) \quad A = A(u_0), \quad B = B(u_0, u_1, \dots, u_{d-1}), \quad D_x(B) - D_t(A) + [A, B] = 0$$

correspond to representations of this algebra (see, e.g., [1, 9, 22]) and references therein). This algebra is called the *Wahlquist-Estabrook prolongation algebra*.

In order to study the general case of ZCRs (2), (3) with arbitrary n , we need to consider gauge transformations.

Without loss of generality, one can assume that \mathfrak{g} is a Lie subalgebra of \mathfrak{gl}_N for some $N \in \mathbb{Z}_{>0}$, where \mathfrak{gl}_N is the algebra of $N \times N$ matrices. Let \mathcal{G} be the connected matrix Lie group corresponding to $\mathfrak{g} \subset \mathfrak{gl}_N$. A *gauge transformation* is given by a function $G = G(x, t, u_0, u_1, \dots, u_m)$ with values in \mathcal{G} .

For any ZCR (2), (3) and any gauge transformation G , the functions

$$(6) \quad \tilde{A} = GAG^{-1} - D_x(G) \cdot G^{-1}, \quad \tilde{B} = GBG^{-1} - D_t(G) \cdot G^{-1}$$

satisfy $D_x(\tilde{B}) - D_t(\tilde{A}) + [\tilde{A}, \tilde{B}] = 0$ and, therefore, form a ZCR. The ZCR given by (6) is said to be *gauge equivalent* to the ZCR (2), (3).

The WE method does not consider gauge transformations. In the classification of ZCRs (5) this is acceptable, because the class of ZCRs (5) is relatively small.

The class of ZCRs (2), (3) is much larger than that of (5). As we show below, gauge transformations play a very important role in the classification of ZCRs (2), (3). Because of this, the classical WE method does not produce satisfactory results for (2), (3).

To overcome this problem, we combine the technique of gauge transformations with ideas similar to the WE method. Loosely speaking, the main ideas can be stated as follows.

We find a normal form for ZCRs (2), (3) with respect to the action of the group of gauge transformations. This allows us to define a Lie algebra \mathbb{F}^n for each $n \in \mathbb{Z}_{\geq 0}$ such that the following property holds. For every Lie algebra \mathfrak{g} , any \mathfrak{g} -valued ZCR (2), (3) of order $\leq n$ is locally gauge equivalent to the ZCR arising from a homomorphism $\mathbb{F}^n \rightarrow \mathfrak{g}$.

More precisely, as is discussed below, we define a Lie algebra \mathbb{F}^n for each $n \in \mathbb{Z}_{\geq 0}$ and each point a of the infinite prolongation \mathcal{E} of equation (1). So the full notation for the algebra is $\mathbb{F}^n(\mathcal{E}, a)$.

Recall that the *infinite prolongation* \mathcal{E} of (1) is the infinite-dimensional manifold with the coordinates x, t, u_k for $k \in \mathbb{Z}_{\geq 0}$. The precise definition of $\mathbb{F}^n(\mathcal{E}, a)$ for any equation (1) is presented in Section 2. In this definition, the algebra $\mathbb{F}^n(\mathcal{E}, a)$ is given in terms of generators and relations.

For every Lie algebra \mathfrak{g} , homomorphisms $\mathbb{F}^n(\mathcal{E}, a) \rightarrow \mathfrak{g}$ classify (up to gauge equivalence) all \mathfrak{g} -valued ZCRs (2), (3) of order $\leq n$, where functions A, B are defined on a neighborhood of the point $a \in \mathcal{E}$. See Section 2 for details.

As we show in Remark 4 below, in some examples the algebras $\mathbb{F}^n(\mathcal{E}, a)$ are more interesting than Wahlquist-Estabrook prolongation algebras.

Some applications of $\mathbb{F}^n(\mathcal{E}, a)$ to classification of evolution equations with respect to Bäcklund transformations are discussed in Subsection 1.3.

Let \mathbb{K} be either \mathbb{C} or \mathbb{R} . We suppose that the variables x, t, u_k take values in \mathbb{K} . A point $a \in \mathcal{E}$ is determined by the values of the coordinates x, t, u_k at a . Let

$$a = (x = x_0, t = t_0, u_k = a_k) \in \mathcal{E}, \quad x_0, t_0, a_k \in \mathbb{K}, \quad k \in \mathbb{Z}_{\geq 0},$$

be a point of \mathcal{E} .

To clarify the definition of $\mathbb{F}^n(\mathcal{E}, a)$, let us consider the case $n = 1$. According to Theorem 6 in Section 2, any ZCR of the form

$$(7) \quad A = A(x, t, u_0, u_1), \quad B = B(x, t, u_0, u_1, \dots, u_d), \quad D_x(B) - D_t(A) + [A, B] = 0$$

on a neighborhood of $a \in \mathcal{E}$ is gauge equivalent to a ZCR of the form

$$(8) \quad \tilde{A} = \tilde{A}(x, t, u_0, u_1), \quad \tilde{B} = \tilde{B}(x, t, u_0, u_1, \dots, u_d),$$

$$(9) \quad D_x(\tilde{B}) - D_t(\tilde{A}) + [\tilde{A}, \tilde{B}] = 0,$$

$$(10) \quad \frac{\partial \tilde{A}}{\partial u_1}(x, t, u_0, a_1) = 0, \quad \tilde{A}(x, t, a_0, a_1) = 0, \quad \tilde{B}(x_0, t, a_0, a_1, \dots, a_d) = 0.$$

In other words, properties (8), (10) determine a normal form for ZCRs (7) with respect to the action of the group of gauge transformations on a neighborhood of $a \in \mathcal{E}$.

A similar normal form for ZCRs (2), (3) with arbitrary n is described in Theorem 6.

As has been said above, all considered functions are assumed to be analytic. Therefore, on a neighborhood of $a \in \mathcal{E}$, the functions \tilde{A}, \tilde{B} from (8), (10) are represented as absolutely convergent power series

$$(11) \quad \tilde{A} = \sum_{l_1, l_2, i_0, i_1 \geq 0} (x - x_0)^{l_1} (t - t_0)^{l_2} (u_0 - a_0)^{i_0} (u_1 - a_1)^{i_1} \cdot \tilde{A}_{i_0, i_1}^{l_1, l_2},$$

$$(12) \quad \tilde{B} = \sum_{l_1, l_2, j_0, \dots, j_d \geq 0} (x - x_0)^{l_1} (t - t_0)^{l_2} (u_0 - a_0)^{j_0} \dots (u_d - a_d)^{j_d} \cdot \tilde{B}_{j_0 \dots j_d}^{l_1, l_2}.$$

Here $\tilde{A}_{i_0, i_1}^{l_1, l_2}$ and $\tilde{B}_{j_0 \dots j_d}^{l_1, l_2}$ are elements of a Lie algebra, which we do not specify yet.

Using formulas (11), (12), we see that properties (10) are equivalent to

$$(13) \quad \tilde{A}_{i_0, 1}^{l_1, l_2} = \tilde{A}_{0, 0}^{l_1, l_2} = \tilde{B}_{0 \dots 0}^{0, l_2} = 0, \quad l_1, l_2, i_0 \in \mathbb{Z}_{\geq 0}.$$

To define $\mathbb{F}^1(\mathcal{E}, a)$, we regard $\tilde{A}_{i_0, i_1}^{l_1, l_2}, \tilde{B}_{j_0 \dots j_d}^{l_1, l_2}$ from (11), (12) as abstract symbols. By definition, the algebra $\mathbb{F}^1(\mathcal{E}, a)$ is generated by the symbols $\tilde{A}_{i_0, i_1}^{l_1, l_2}, \tilde{B}_{j_0 \dots j_d}^{l_1, l_2}$ for $l_1, l_2, i_0, i_1, j_0, \dots, j_d \in \mathbb{Z}_{\geq 0}$. Relations for these generators are provided by equations (9), (13). A more detailed description of this construction is given in Section 2.

According to Section 2, the algebras $\mathbb{F}^n(\mathcal{E}, a)$ for $n \in \mathbb{Z}_{\geq 0}$ are arranged in a sequence of surjective homomorphisms

$$(14) \quad \dots \rightarrow \mathbb{F}^n(\mathcal{E}, a) \rightarrow \mathbb{F}^{n-1}(\mathcal{E}, a) \rightarrow \dots \rightarrow \mathbb{F}^1(\mathcal{E}, a) \rightarrow \mathbb{F}^0(\mathcal{E}, a).$$

Remark 1. It is possible to introduce an analog of $\mathbb{F}^n(\mathcal{E}, a)$ for evolution systems

$$\begin{aligned} \frac{\partial u^i}{\partial t} &= F^i(x, t, u^1, \dots, u^m, u_1^1, \dots, u_1^m, \dots, u_d^1, \dots, u_d^m), \\ u^i &= u^i(x, t), \quad u_k^i = \frac{\partial^k u^i}{\partial x^k}, \quad i = 1, \dots, m. \end{aligned}$$

Some results in this direction are sketched in the preprint [8].

In the present paper we study only the scalar case $m = 1$, because for scalar evolution equations the structure of $\mathbb{F}^n(\mathcal{E}, a)$ can be understood much better. For $m > 1$ one gets some interesting results as well (see [8]), but the case of $m > 1$ is significantly different from that of $m = 1$.

As has been said above, the algebra $\mathbb{F}^n(\mathcal{E}, a)$ is defined by a certain set of generators and relations arising from a normal form of ZCRs. In Proposition 5 in Section 3 we describe a smaller subset of generators for $\mathbb{F}^n(\mathcal{E}, a)$.

Example 1. Consider the case $n = 1$. As has been said above, the algebra $\mathbb{F}^1(\mathcal{E}, a)$ is given by the generators $\tilde{A}_{i_0, i_1}^{l_1, l_2}$, $\tilde{B}_{j_0 \dots j_d}^{l_1, l_2}$ and the relations arising from (9), (13). Proposition 5 implies that the algebra $\mathbb{F}^1(\mathcal{E}, a)$ coincides with the subalgebra generated by $\tilde{A}_{i_0, i_1}^{l_1, 0}$ for $l_1, i_0, i_1 \in \mathbb{Z}_{\geq 0}$.

According to Proposition 5, a similar result is valid also for $\mathbb{F}^n(\mathcal{E}, a)$ for any n .

This result helps us to describe the structure of $\mathbb{F}^n(\mathcal{E}, a)$ and the homomorphisms (14) more explicitly for some PDEs. Let $q \in \mathbb{Z}_{>0}$. Consider equations of the form

$$(15) \quad u_t = u_{2q+1} + f(x, t, u_0, u_1, \dots, u_{2q-1}).$$

This class of PDEs includes the KdV, Kaup-Kupershmidt, Sawada-Kotera equations.

Theorems 1 and 2 are proved in Sections 4 and 6 respectively.

Theorem 1 (Section 4). *Let \mathcal{E} be the infinite prolongation of equation (15), where f is an arbitrary function and $q \in \mathbb{Z}_{>0}$. Let $a \in \mathcal{E}$. For each $n \in \mathbb{Z}_{>0}$, consider the homomorphism $\varphi_n: \mathbb{F}^n(\mathcal{E}, a) \rightarrow \mathbb{F}^{n-1}(\mathcal{E}, a)$ from (14).*

If $n \geq 2q - 1$ then

$$[v_1, v_2] = 0 \quad \forall v_1 \in \ker \varphi_n, \quad \forall v_2 \in \mathbb{F}^n(\mathcal{E}, a).$$

In other words, if $n \geq 2q - 1$ then the kernel of φ_n is contained in the center of the Lie algebra $\mathbb{F}^n(\mathcal{E}, a)$.

For each $k \in \mathbb{Z}_{>0}$, let $\psi_k: \mathbb{F}^{k+2q-2}(\mathcal{E}, a) \rightarrow \mathbb{F}^{2q-2}(\mathcal{E}, a)$ be the composition of the homomorphisms

$$\mathbb{F}^{k+2q-2}(\mathcal{E}, a) \rightarrow \mathbb{F}^{k+2q-3}(\mathcal{E}, a) \rightarrow \dots \rightarrow \mathbb{F}^{2q-1}(\mathcal{E}, a) \rightarrow \mathbb{F}^{2q-2}(\mathcal{E}, a)$$

from (14). Then

$$[h_1, [h_2, \dots, [h_{k-1}, [h_k, h_{k+1}]] \dots]] = 0 \quad \forall h_1, \dots, h_{k+1} \in \ker \psi_k.$$

In particular, the kernel of ψ_k is nilpotent.

Applications of Theorem 1 to obtaining necessary conditions for integrability of equations (15) are discussed in Subsection 1.2.

Theorem 2 (Section 6). *Consider the infinite-dimensional Lie algebra*

$$\mathfrak{sl}_2(\mathbb{K}[\lambda]) \cong \mathfrak{sl}_2(\mathbb{K}) \otimes_{\mathbb{K}} \mathbb{K}[\lambda],$$

where $\mathbb{K}[\lambda]$ is the algebra of polynomials in λ . Let \mathcal{E} be the infinite prolongation of the KdV equation

$$(16) \quad u_t = u_{xxx} + u_x u.$$

Let $a \in \mathcal{E}$. Then

- *the algebra $\mathbb{F}^0(\mathcal{E}, a)$ is isomorphic to the direct sum of $\mathfrak{sl}_2(\mathbb{K}[\lambda])$ and a 3-dimensional abelian Lie algebra,*
- *for each $n \in \mathbb{Z}_{>0}$, the kernel of the surjective homomorphism $\mathbb{F}^n(\mathcal{E}, a) \rightarrow \mathbb{F}^0(\mathcal{E}, a)$ from (14) is nilpotent.*

To describe $\mathbb{F}^0(\mathcal{E}, a)$ for the KdV equation in Theorem 2, we use the following fact. If the function F in (1) does not depend on x, t , then the algebra $\mathbb{F}^0(\mathcal{E}, a)$ is isomorphic to a certain subalgebra of the Wahlquist-Estabrook prolongation algebra for (1) (see Theorem 9 in Section 5 for details).

The explicit structure of the Wahlquist-Estabrook prolongation algebra for the KdV equation is given in [2, 3], and this allows us to describe $\mathbb{F}^0(\mathcal{E}, a)$ for the KdV equation.

Remark 2. Using some extra computations, one can prove the following.

Proposition 1. *Let \mathcal{E} be the infinite prolongation of the KdV equation. For any $a \in \mathcal{E}$ and any $n \in \mathbb{Z}_{>0}$, the algebra $\mathbb{F}^n(\mathcal{E}, a)$ is isomorphic to the direct sum of $\mathfrak{sl}_2(\mathbb{K}[\lambda])$ and a finite-dimensional nilpotent Lie algebra.*

We do not present a proof of Proposition 1 in this paper, because the result of Theorem 2 is sufficient for the main applications to Bäcklund transformations, which are discussed in Subsection 1.3.

To describe another example, we need some auxiliary constructions. Let $\mathbb{K}[v_1, v_2, v_3]$ be the algebra of polynomials in the variables v_1, v_2, v_3 . Let $e_1, e_2, e_3 \in \mathbb{K}$ be such that $e_1 \neq e_2 \neq e_3 \neq e_1$.

Consider the ideal $\mathcal{I}_{e_1, e_2, e_3} \subset \mathbb{K}[v_1, v_2, v_3]$ generated by the polynomials

$$(17) \quad v_i^2 - v_j^2 + e_i - e_j, \quad i, j = 1, 2, 3.$$

Set

$$E_{e_1, e_2, e_3} = \mathbb{K}[v_1, v_2, v_3] / \mathcal{I}_{e_1, e_2, e_3}.$$

In other words, E_{e_1, e_2, e_3} is the commutative associative algebra of regular functions on the algebraic curve in \mathbb{K}^3 defined by the polynomials (17). It is easy to check that this curve is nonsingular and is of genus 1.

We have the natural surjective homomorphism $\mathbb{K}[v_1, v_2, v_3] \rightarrow E_{e_1, e_2, e_3}$. The image of $v_i \in \mathbb{K}[v_1, v_2, v_3]$ in E_{e_1, e_2, e_3} is denoted by $\bar{v}_i \in E_{e_1, e_2, e_3}$ for $i = 1, 2, 3$.

Consider also a basis x_1, x_2, x_3 of the Lie algebra $\mathfrak{so}_3(\mathbb{K})$ such that

$$[x_1, x_2] = x_3, \quad [x_2, x_3] = x_1, \quad [x_3, x_1] = x_2.$$

We endow the space $\mathfrak{so}_3(\mathbb{K}) \otimes_{\mathbb{K}} E_{e_1, e_2, e_3}$ with the following Lie algebra structure

$$[y_1 \otimes h_1, y_2 \otimes h_2] = [y_1, y_2] \otimes h_1 h_2, \quad y_1, y_2 \in \mathfrak{so}_3(\mathbb{K}), \quad h_1, h_2 \in E_{e_1, e_2, e_3}.$$

Denote by $\mathfrak{R}_{e_1, e_2, e_3}$ the Lie subalgebra of $\mathfrak{so}_3(\mathbb{K}) \otimes_{\mathbb{K}} E_{e_1, e_2, e_3}$ generated by the elements

$$x_i \otimes \bar{v}_i \in \mathfrak{so}_3(\mathbb{K}) \otimes_{\mathbb{K}} E_{e_1, e_2, e_3}, \quad i = 1, 2, 3.$$

It is easily seen that the Lie algebra $\mathfrak{R}_{e_1, e_2, e_3}$ is infinite-dimensional. According to [17], the Wahlquist-Estabrook prolongation algebra of the anisotropic Landau-Lifshitz equation is isomorphic to the direct sum of $\mathfrak{R}_{e_1, e_2, e_3}$ and a 2-dimensional abelian Lie algebra.

According to Proposition 2 below, the algebra $\mathfrak{R}_{e_1, e_2, e_3}$ appears also in the structure of the algebras $\mathbb{F}^n(\mathcal{E}, a)$ for the Krichever-Novikov equation. A proof of Proposition 2 in the case $\mathbb{K} = \mathbb{C}$ is sketched in [7].

Proposition 2 ([7]). *For any $e_1, e_2, e_3 \in \mathbb{K}$, consider the Krichever-Novikov equation*

$$(18) \quad u_t = u_{xxx} - \frac{3}{2} \frac{u_{xx}^2}{u_x} + \frac{(u - e_1)(u - e_2)(u - e_3)}{u_x}, \quad u = u(x, t).$$

Let \mathcal{E} be the infinite prolongation of this equation. Let $a \in \mathcal{E}$. Then

- *the algebra $\mathbb{F}^0(\mathcal{E}, a)$ is zero,*
- *for any $n \geq 2$, the kernel of the surjective homomorphism $\mathbb{F}^n(\mathcal{E}, a) \rightarrow \mathbb{F}^1(\mathcal{E}, a)$ from (14) is nilpotent,*
- *if $e_1 \neq e_2 \neq e_3 \neq e_1$, then the algebra $\mathbb{F}^1(\mathcal{E}, a)$ is isomorphic to $\mathfrak{R}_{e_1, e_2, e_3}$.*

Remark 3. The proof of Proposition 2 uses the well-known fact that the Krichever-Novikov equation possesses an \mathfrak{so}_3 -valued zero-curvature representation parametrized by the above-mentioned curve.

Remark 4. As has been said above, if the function F in (1) does not depend on x, t , then the algebra $\mathbb{F}^0(\mathcal{E}, a)$ is isomorphic to a certain subalgebra of the Wahlquist-Estabrook prolongation algebra for (1).

The algebras $\mathbb{F}^n(\mathcal{E}, a)$ for $n \geq 1$ cannot be obtained by the classical Wahlquist-Estabrook prolongation method, because the main idea behind the definition of $\mathbb{F}^n(\mathcal{E}, a)$ is based on the use of gauge transformations, while the Wahlquist-Estabrook prolongation method does not consider gauge transformations.

According to Proposition 2, for the Krichever-Novikov equation we have $\mathbb{F}^0(\mathcal{E}, a) = 0$ and $\dim \mathbb{F}^n(\mathcal{E}, a) = \infty$ for $n \geq 1$. It is easy to show that the classical Wahlquist-Estabrook prolongation algebra is trivial for the Krichever-Novikov equation. Thus in this example the algebras $\mathbb{F}^n(\mathcal{E}, a)$ are more interesting than the Wahlquist-Estabrook prolongation algebra.

Remark 5. For the KdV, Krichever-Novikov equations and the equation $u_t = u_{xxx}$, the problem to describe ZCRs the form

$$(19) \quad A = A(u_0, u_1, u_2, \dots), \quad B = B(u_0, u_1, u_2, \dots), \quad D_x(B) - D_t(A) + [A, B] = 0$$

was studied in [5]. Note that A, B in (19) do not depend on x, t .

For the Burgers and KdV equations, the problem to describe ZCRs of the form (19) was also studied in [4]. However, gauge transformations were not considered in [4]. Because of this, the paper [4] had to impose some additional constraints on the functions A, B in (19).

Remark 6. Some other approaches to the action of gauge transformations on ZCRs can be found in [10, 11, 12, 18, 19, 20] and references therein. For a given \mathfrak{g} -valued ZCR, the papers [10, 11, 18] define certain \mathfrak{g} -valued functions that transform by conjugation when the ZCR transforms by gauge. Applications of these functions to construction and classification of some types of ZCRs are described in [10, 11, 12, 18, 19, 20] and references therein.

To our knowledge, the theory of [10, 11, 12, 18, 19, 20] does not lead to any infinite-dimensional Lie algebras responsible for ZCRs. So this theory does not contain the algebras $\mathbb{F}^n(\mathcal{E}, a)$.

1.2. Necessary conditions for integrability. In this subsection, \mathfrak{g} is a finite-dimensional matrix Lie algebra, and \mathcal{E} is the infinite prolongation of an equation of the form (20). ZCRs and gauge transformations are supposed to be defined on a neighborhood of a point $a \in \mathcal{E}$.

Theorem 1 implies the following.

Theorem 3. *Let \mathcal{E} be the infinite prolongation of an equation of the form*

$$(20) \quad u_t = u_{2q+1} + f(x, t, u_0, u_1, \dots, u_{2q-1}), \quad q \in \mathbb{Z}_{>0}.$$

Let $a \in \mathcal{E}$.

If the Lie algebra $\mathbb{F}^{2q-2}(\mathcal{E}, a)$ is nilpotent, then $\mathbb{F}^n(\mathcal{E}, a)$ is also nilpotent for all $n \geq 2q - 2$.

If $\mathbb{F}^{2q-2}(\mathcal{E}, a)$ is solvable, then $\mathbb{F}^n(\mathcal{E}, a)$ is solvable for all $n \geq 2q - 2$.

According to Theorem 7 in Section 2, for any \mathfrak{g} -valued ZCR of order $\leq n$, there is a homomorphism $\rho: \mathbb{F}^n(\mathcal{E}, a) \rightarrow \mathfrak{g}$ such that this ZCR is gauge equivalent to a ZCR with values in the Lie subalgebra $\rho(\mathbb{F}^n(\mathcal{E}, a)) \subset \mathfrak{g}$. Using this fact and Theorem 3, we get the following result.

Theorem 4. *Let \mathcal{E} be the infinite prolongation of an equation of the form (20). Let $a \in \mathcal{E}$.*

If $\mathbb{F}^{2q-2}(\mathcal{E}, a)$ is nilpotent, then, for every Lie algebra \mathfrak{g} , any \mathfrak{g} -valued ZCR

$$(21) \quad A = A(x, t, u_0, u_1, \dots, u_n), \quad B = B(x, t, u_0, u_1, \dots, u_{n+2q}), \\ D_x(B) - D_t(A) + [A, B] = 0.$$

is gauge equivalent to a ZCR with values in a nilpotent Lie subalgebra of \mathfrak{g} .

If $\mathbb{F}^{2q-2}(\mathcal{E}, a)$ is solvable, then any \mathfrak{g} -valued ZCR (21) is gauge equivalent to a ZCR with values in a solvable Lie subalgebra of \mathfrak{g} .

Recall that \mathfrak{g} is a finite-dimensional matrix Lie algebra. Let \mathcal{G} be the connected matrix Lie group corresponding to \mathfrak{g} . As has been discussed in Section 1.1, a gauge transformation is given by a function $G = G(x, t, u_0, u_1, \dots, u_k)$ with values in \mathcal{G} .

A \mathfrak{g} -valued ZCR

$$A = A(x, t, u_0, u_1, \dots), \quad B = B(x, t, u_0, u_1, \dots), \quad D_x(B) - D_t(A) + [A, B] = 0$$

is called *gauge-solvable* if there is a gauge transformation G such that the functions

$$\tilde{A} = GAG^{-1} - D_x(G) \cdot G^{-1}, \quad \tilde{B} = GBG^{-1} - D_t(G) \cdot G^{-1}$$

take values in a solvable Lie subalgebra of \mathfrak{g} . In other words, a \mathfrak{g} -valued ZCR is gauge-solvable iff it is gauge equivalent to a ZCR with values in a solvable Lie subalgebra of \mathfrak{g} .

Recall that, in this paper, integrability of PDEs is understood in the sense of soliton theory and the inverse scattering method.¹ In soliton theory, one is interested in ZCRs that are not gauge-solvable.

To our knowledge, for all known examples of integrable equations (1), one has the following property. If an equation (1) is integrable then it possesses a ZCR that is not gauge-solvable. Therefore, it is natural to take this property as a necessary condition² for integrability of (1).

Combining this observation with Theorem 4, we obtain the following.

Theorem 5. *Let \mathcal{E} be the infinite prolongation of an equation of the form (20). If the Lie algebra $\mathbb{F}^{2q-2}(\mathcal{E}, a)$ is solvable for all $a \in \mathcal{E}$, then this equation is not integrable in the sense of soliton theory.*

In other words, the property

$$(22) \quad \text{the Lie algebra } \mathbb{F}^{2q-2}(\mathcal{E}, a) \text{ is not solvable for some } a \in \mathcal{E}$$

is a necessary condition for integrability of equations of the form (20).

Example 2. To clarify the above results, let us consider a simple example in the case $q = 1$. Consider the equation

$$(23) \quad u_t = u_{xxx} + u^4.$$

It is easy to check that this equation does not admit any nontrivial ZCRs (2), (3) for $n = 0$. However, it is not clear whether equation (23) possesses nontrivial ZCRs (2), (3) for $n > 0$ which could in principle be used for establishing integrability of (23).

Using the definition of $\mathbb{F}^0(\mathcal{E}, a)$ given in Section 2, it is easy to show that $\mathbb{F}^0(\mathcal{E}, a) = 0$ for equation (23). Then Theorem 4 implies that, for all $n \geq 0$, any ZCR (2), (3) of (23) is gauge equivalent to a ZCR with values in a nilpotent Lie algebra. Therefore, we obtain that equation (23) is not integrable.

Remark 7. It is widely believed that the following property holds for all integrable equations. If an equation (1) is integrable then there are a semisimple Lie algebra \mathfrak{g} and a \mathfrak{g} -valued ZCR

$$(24) \quad A = A(\lambda, x, t, u_0, u_1, \dots, u_n), \quad B = B(\lambda, x, t, u_0, u_1, \dots, u_{n+d-1}), \\ D_x(B) - D_t(A) + [A, B] = 0$$

such that A, B depend (analytically) on a parameter λ , which cannot be removed by gauge transformations.

Let \mathfrak{g}^λ be the infinite-dimensional Lie algebra of analytic functions $h(\lambda)$ with values in \mathfrak{g} . Then (24) can be regarded as a ZCR with values in \mathfrak{g}^λ .

It can be shown that the algebras $\mathbb{F}^n(\mathcal{E}, a)$ are responsible also for ZCRs with values in infinite-dimensional Lie algebras. In particular, the ZCR (24) is gauge equivalent to the ZCR determined by a homomorphism $\rho: \mathbb{F}^n(\mathcal{E}, a) \rightarrow \mathfrak{g}^\lambda$.

Using this construction and Theorem 1, one can show that the property

$$(25) \quad \exists a \in \mathcal{E} \quad \dim \mathbb{F}^{2q-2}(\mathcal{E}, a) = \infty$$

is a necessary condition for integrability of equations of the form (20). A proof of this fact will be presented elsewhere.

¹It is well known that linear equations are not integrable in this sense.

²This condition is necessary, but is probably not sufficient for integrability of (1). Another necessary condition can be obtained in the consideration of parameter-dependent ZCRs, which are discussed in Remark 7.

So (22) and (25) are necessary conditions for integrability of equations of the form (20). According to Section 2, the algebra $\mathbb{F}^{2q-2}(\mathcal{E}, a)$ is defined in terms of generators and relations for any given equation (20). For some types of equations (20), it is possible to deduce from (22), (25) some explicit conditions on the function f in (20), but this will be described elsewhere.

Recall that in this paper we study integrability by means of ZCRs. Another well-known approach to integrability of evolution equations involves symmetries and consevation laws (see, e.g., [13, 14, 15, 16, 21] and references therein). In particular, the paper [13] presents necessary conditions for existence of generalized (higher) symmetries and conservation laws for equations of the form

$$(26) \quad u_t = u_d + g(x, u_0, u_1, u_2, \dots, u_{d-1}).$$

Using these conditions, one obtains classification results for some types of integrable equations of the form (26) in the cases $d = 2, 3, 5$ (see [13] and references therein). It would be interesting to compare (22), (25) with the conditions presented in [13].

1.3. Necessary conditions for existence of Bäcklund transformations. The algebras $\mathbb{F}^n(\mathcal{E}, a)$ help also to obtain necessary conditions for existence of a Bäcklund transformation between two given evolution equations. In the present paper we do not study Bäcklund transformations. We just briefly mention some results in this direction.

For each $n \in \mathbb{Z}_{>0}$, consider the surjective homomorphism $\mathbb{F}^n(\mathcal{E}, a) \rightarrow \mathbb{F}^{n-1}(\mathcal{E}, a)$ from (14). Denote this homomorphism by $\mu_n: \mathbb{F}^n(\mathcal{E}, a) \rightarrow \mathbb{F}^{n-1}(\mathcal{E}, a)$.

Let $\mathbb{F}(\mathcal{E}, a)$ be the inverse (projective) limit of the sequence (14). An element of $\mathbb{F}(\mathcal{E}, a)$ is given by a sequence

$$(c_0, c_1, c_2, c_3, \dots), \quad c_n \in \mathbb{F}^n(\mathcal{E}, a), \quad \mu_n(c_n) = c_{n-1}.$$

Then $\mathbb{F}(\mathcal{E}, a)$ is a Lie algebra, and we can consider the following topology on $\mathbb{F}(\mathcal{E}, a)$.

Since $\mathbb{F}(\mathcal{E}, a)$ is the inverse limit of (14), for each $k \in \mathbb{Z}_{\geq 0}$ we have the surjective homomorphism $\rho_k: \mathbb{F}(\mathcal{E}, a) \rightarrow \mathbb{F}^k(\mathcal{E}, a)$ given by $\rho_k((c_0, c_1, c_2, c_3, \dots)) = c_k$.

The subsets $\rho_k^{-1}(v) \subset \mathbb{F}(\mathcal{E}, a)$ for $v \in \mathbb{F}^k(\mathcal{E}, a)$ and $k \in \mathbb{Z}_{\geq 0}$ form a base of the topology on $\mathbb{F}(\mathcal{E}, a)$.

Remark 8. Let L be a Lie algebra endowed with the discrete topology. Then a homomorphism $\mathbb{F}(\mathcal{E}, a) \rightarrow L$ is continuous iff it is of the form $\mathbb{F}(\mathcal{E}, a) \xrightarrow{\rho_k} \mathbb{F}^k(\mathcal{E}, a) \rightarrow L$ for some $k \in \mathbb{Z}_{\geq 0}$ and some homomorphism $\mathbb{F}^k(\mathcal{E}, a) \rightarrow L$.

It is shown in [7] that the algebra $\mathbb{F}(\mathcal{E}, a)$ has some coordinate-independent geometric meaning.

A Lie subalgebra $H \subset \mathbb{F}(\mathcal{E}, a)$ is said to be *tame* if there are $k \in \mathbb{Z}_{\geq 0}$ and a subalgebra $\mathfrak{h} \subset \mathbb{F}^k(\mathcal{E}, a)$ such that $H = \rho_k^{-1}(\mathfrak{h})$. Note that the codimension of H in $\mathbb{F}(\mathcal{E}, a)$ is equal to the codimension of \mathfrak{h} in $\mathbb{F}^k(\mathcal{E}, a)$.

Remark 9. It is easily seen that a subalgebra $H \subset \mathbb{F}(\mathcal{E}, a)$ is tame iff H is open and closed in $\mathbb{F}(\mathcal{E}, a)$ with respect to the topology on $\mathbb{F}(\mathcal{E}, a)$.

A proof of Proposition 3 is sketched in [7].

Proposition 3 ([7]). *Let \mathcal{E}_1 and \mathcal{E}_2 be evolution PDEs. Suppose that \mathcal{E}_1 and \mathcal{E}_2 are connected by a Bäcklund transformation. Then for each $i = 1, 2$ there is a point $a_i \in \mathcal{E}_i$ and a tame subalgebra $H_i \subset \mathbb{F}(\mathcal{E}_i, a_i)$ such that*

- H_i is of finite codimension in $\mathbb{F}(\mathcal{E}_i, a_i)$,
- H_1 is isomorphic to H_2 , and this isomorphism is a homeomorphism with respect to the topology induced by the embedding $H_i \subset \mathbb{F}(\mathcal{E}_i, a_i)$.

In fact the preprint [7] contains a more general result about PDEs that are not necessarily evolution.

Proposition 3 provides a necessary condition for two given evolution PDEs to be connected by a Bäcklund transformation (BT for short). Using Proposition 3, one can prove non-existence of BTs for some PDEs.

For example, the following result is obtained in [6] by means of this theory. For any $e_1, e_2, e_3 \in \mathbb{C}$, consider the Krichever-Novikov equation

$$(27) \quad \text{KN}(e_1, e_2, e_3) = \left\{ u_t = u_{xxx} - \frac{3}{2} \frac{u_{xx}^2}{u_x} + \frac{(u - e_1)(u - e_2)(u - e_3)}{u_x}, \quad u = u(x, t) \right\}$$

and the algebraic curve $C(e_1, e_2, e_3) = \left\{ (z, y) \in \mathbb{C}^2 \mid y^2 = (z - e_1)(z - e_2)(z - e_3) \right\}$.

Proposition 4 ([6]). *Let $e_1, e_2, e_3, e'_1, e'_2, e'_3 \in \mathbb{C}$ be such that $e_i \neq e_j$ and $e'_i \neq e'_j$ for all $i \neq j$.*

If the curve $C(e_1, e_2, e_3)$ is not birationally equivalent to the curve $C(e'_1, e'_2, e'_3)$, then the equation $\text{KN}(e_1, e_2, e_3)$ is not connected with the equation $\text{KN}(e'_1, e'_2, e'_3)$ by any Bäcklund transformation.

Also, if $e_1 \neq e_2 \neq e_3 \neq e_1$, then $\text{KN}(e_1, e_2, e_3)$ is not connected with the KdV equation by any BT.

BTs of Miura type (differential substitutions) for (27) were studied in [21]. According to [21], the equation $\text{KN}(e_1, e_2, e_3)$ is connected with the KdV equation by a BT of Miura type iff $e_i = e_j$ for some $i \neq j$.

The preprints [6, 7] and Propositions 3, 4 consider the most general class of BTs, which is much larger than the class of BTs of Miura type studied in [21].

Note that the present paper is self-contained and can be studied independently of [6, 7].

1.4. Abbreviations, conventions, and notation. The following abbreviations, conventions, and notation are used in the paper. ZCR = zero-curvature representation, WE = Wahlquist-Estabrook, BT = Bäcklund transformation.

All manifolds and functions are supposed to be analytic.

The symbols $\mathbb{Z}_{>0}$ and $\mathbb{Z}_{\geq 0}$ denote the sets of positive and nonnegative integers respectively.

\mathbb{K} is either \mathbb{C} or \mathbb{R} . All vector spaces and algebras are supposed to be over the field \mathbb{K} .

2. THE ALGEBRAS $\mathbb{F}^n(\mathcal{E}, a)$

Recall that x, t, u_k take values in \mathbb{K} , where \mathbb{K} is either \mathbb{C} or \mathbb{R} . Let \mathbb{K}^∞ be the infinite-dimensional space with the coordinates x, t, u_k for $k \in \mathbb{Z}_{\geq 0}$. The topology on \mathbb{K}^∞ is defined as follows.

For each $l \in \mathbb{Z}_{\geq 0}$, consider the space \mathbb{K}^{l+3} with the coordinates x, t, u_k for $k \leq l$. One has the natural projection $\pi_l: \mathbb{K}^\infty \rightarrow \mathbb{K}^{l+3}$ that “forgets” the coordinates $u_{k'}$ for $k' > l$.

We consider the standard topology on \mathbb{K}^{l+3} . For any $l \in \mathbb{Z}_{\geq 0}$ and any open subset $V \subset \mathbb{K}^{l+3}$, the preimage $\pi_l^{-1}(V) \subset \mathbb{K}^\infty$ is, by definition, open in \mathbb{K}^∞ . Such subsets form a base of the topology on \mathbb{K}^∞ . In other words, we consider the smallest topology on \mathbb{K}^∞ such that all the maps π_l are continuous.

Let $U \subset \mathbb{K}^{d+3}$ be an open subset such that the function $F(x, t, u_0, u_1, \dots, u_d)$ from (1) is defined on U . The *infinite prolongation* \mathcal{E} of equation (1) can be defined as follows $\mathcal{E} = \pi_d^{-1}(U) \subset \mathbb{K}^\infty$.

So \mathcal{E} is an open subset of the space \mathbb{K}^∞ with the coordinates x, t, u_k for $k \in \mathbb{Z}_{\geq 0}$. The topology on \mathcal{E} is induced by the embedding $\mathcal{E} \subset \mathbb{K}^\infty$.

A point $a \in \mathcal{E}$ is determined by the values of the coordinates x, t, u_k at a . Let

$$(28) \quad a = (x = x_0, t = t_0, u_k = a_k) \in \mathcal{E}, \quad x_0, t_0, a_k \in \mathbb{K}, \quad k \in \mathbb{Z}_{\geq 0},$$

be a point of \mathcal{E} .

Let $s \in \mathbb{Z}_{\geq 0}$. For a function $M(x, t, u_0, u_1, u_2, \dots)$, the notation

$$M \Big|_{u_k = a_k, k \geq s}$$

means that we substitute $u_k = a_k$ for all $k \geq s$ in the function M .

Also, sometimes we need to substitute $x = x_0$ or $t = t_0$ in such functions. For example, if $M = M(x, t, u_0, u_1, u_2, u_3)$, then

$$M \Big|_{x=x_0, u_k=a_k, k \geq 2} = M(x_0, t, u_0, u_1, a_2, a_3).$$

Let \mathfrak{gl}_N be the algebra of $N \times N$ matrices with entries from \mathbb{K} . Denote by $\text{Id} \in \mathfrak{gl}_N$ the identity matrix.

Theorem 6. *Let $N \in \mathbb{Z}_{>0}$. Let $\mathfrak{g} \subset \mathfrak{gl}_N$ be a matrix Lie algebra. Denote by \mathcal{G} the connected matrix Lie group corresponding to $\mathfrak{g} \subset \mathfrak{gl}_N$.*

Let

$$(29) \quad A = A(x, t, u_0, u_1, \dots, u_n), \quad B = B(x, t, u_0, u_1, \dots, u_{n+d-1}),$$

$$(30) \quad D_x(B) - D_t(A) + [A, B] = 0$$

be a ZCR of order $\leq n$ such that the functions A, B are defined on a neighborhood of $a \in \mathcal{E}$ and take values in \mathfrak{g} .

Then there is a \mathcal{G} -valued function $G = G(x, t, u_0, u_1, \dots, u_{n-1})$ on a neighborhood of $a \in \mathcal{E}$ such that the functions

$$(31) \quad \tilde{A} = GAG^{-1} - D_x(G) \cdot G^{-1}, \quad \tilde{B} = GBG^{-1} - D_t(G) \cdot G^{-1}$$

satisfy

$$(32) \quad \frac{\partial \tilde{A}}{\partial u_s} \Big|_{u_k=a_k, k \geq s} = 0 \quad \forall s \geq 1,$$

$$(33) \quad \tilde{A} \Big|_{u_k=a_k, k \geq 0} = 0,$$

$$(34) \quad \tilde{B} \Big|_{x=x_0, u_k=a_k, k \geq 0} = 0.$$

Proof. To explain the main idea, let us consider first the case $n = 2$. So $A = A(x, t, u_0, u_1, u_2)$.

Consider the ordinary differential equation (ODE)

$$(35) \quad \frac{\partial G_1}{\partial u_1} = G_1 \cdot \left(\frac{\partial A}{\partial u_2} \Big|_{u_k=a_k, k \geq 2} \right)$$

with respect to the variable u_1 and an unknown function $G_1 = G_1(x, t, u_0, u_1)$. The variables x, t, u_0 are regarded as parameters in this ODE.

Let $G_1(x, t, u_0, u_1)$ be a local solution of the ODE (35) with the initial condition $G_1(x, t, u_0, a_1) = \text{Id}$. Since $\partial A / \partial u_2$ takes values in \mathfrak{g} , the function $G_1(x, t, u_0, u_1)$ takes values in \mathcal{G} .

Set

$$(36) \quad \hat{A} = G_1 A G_1^{-1} - D_x(G_1) \cdot G_1^{-1}, \quad \hat{B} = G_1 B G_1^{-1} - D_t(G_1) \cdot G_1^{-1}.$$

Since G_1 takes values in \mathcal{G} , the functions \hat{A}, \hat{B} take values in \mathfrak{g} . We have also

$$\frac{\partial \hat{A}}{\partial u_2} \Big|_{u_k=a_k, k \geq 2} = 0.$$

Now consider the ODE

$$(37) \quad \frac{\partial G_0}{\partial u_0} = G_0 \cdot \left(\frac{\partial \hat{A}}{\partial u_1} \Big|_{u_k=a_k, k \geq 1} \right)$$

with respect to the variable u_0 and an unknown function $G_0 = G_0(x, t, u_0)$, where x, t are regarded as parameters.

Let $G_0(x, t, u_0)$ be a local solution of the ODE (37) with the initial condition $G_0(x, t, a_0) = \text{Id}$. Let

$$(38) \quad \bar{A} = G_0 \hat{A} G_0^{-1} - D_x(G_0) \cdot G_0^{-1}, \quad \bar{B} = G_0 \hat{B} G_0^{-1} - D_t(G_0) \cdot G_0^{-1}.$$

Then

$$\left. \frac{\partial \bar{A}}{\partial u_s} \right|_{u_k=a_k, k \geq s} = 0 \quad \forall s \geq 1.$$

Let $\tilde{G} = \tilde{G}(x, t)$ be a local solution of the ODE

$$\frac{\partial \tilde{G}}{\partial x} = \tilde{G} \cdot \left(\bar{A} \Big|_{u_k=a_k, k \geq 0} \right)$$

with the initial condition $\tilde{G}(x_0, t) = \text{Id}$, where t is regarded as a parameter. Set

$$(39) \quad \check{A} = \tilde{G} \bar{A} \tilde{G}^{-1} - D_x(\tilde{G}) \cdot \tilde{G}^{-1}, \quad \check{B} = \tilde{G} \bar{B} \tilde{G}^{-1} - D_t(\tilde{G}) \cdot \tilde{G}^{-1}.$$

Then

$$\left. \frac{\partial \check{A}}{\partial u_s} \right|_{u_k=a_k, k \geq s} = 0 \quad \forall s \geq 1, \quad \check{A} \Big|_{u_k=a_k, k \geq 0} = 0.$$

Finally, let $\hat{G} = \hat{G}(t)$ be a local solution of the ODE

$$(40) \quad \frac{\partial \hat{G}}{\partial t} = \hat{G} \cdot \left(\check{B} \Big|_{x=x_0, u_k=a_k, k \geq 0} \right)$$

with the initial condition $\hat{G}(t_0) = \text{Id}$. Let

$$(41) \quad \tilde{A} = \hat{G} \check{A} \hat{G}^{-1} - D_x(\hat{G}) \cdot \hat{G}^{-1}, \quad \tilde{B} = \hat{G} \check{B} \hat{G}^{-1} - D_t(\hat{G}) \cdot \hat{G}^{-1}.$$

Then \tilde{A}, \tilde{B} obey (32), (33), (34).

Set $G = \hat{G} \cdot \tilde{G} \cdot G_0 \cdot G_1$. Then equations (36), (38), (39), (41) imply

$$\tilde{A} = G A G^{-1} - D_x(G) \cdot G^{-1}, \quad \tilde{B} = G B G^{-1} - D_t(G) \cdot G^{-1}.$$

Therefore, $G = \hat{G} \cdot \tilde{G} \cdot G_0 \cdot G_1$ satisfies all the required properties in the case $n = 2$.

This construction can be easily generalized to the case of arbitrary n . One can define G as the product

$$G = \hat{G} \cdot \tilde{G} \cdot G_0 \cdot G_1 \dots G_{n-1},$$

where the functions

$$G_q = G_q(x, t, u_0, \dots, u_q), \quad q = 0, 1, \dots, n-1, \\ \tilde{G} = \tilde{G}(x, t), \quad \hat{G} = \hat{G}(t).$$

are defined as solutions of certain ODEs similar to the ODEs considered above. \square

Remark 10. Since the functions (29) obey (30), the functions (31) satisfy

$$(42) \quad D_x(\tilde{B}) - D_t(\tilde{A}) + [\tilde{A}, \tilde{B}] = 0.$$

Recall that all functions are supposed to be analytic. Therefore, taking a sufficiently small neighborhood of $a \in \mathcal{E}$, we can assume that \tilde{A}, \tilde{B} are represented as absolutely convergent power series

$$(43) \quad \tilde{A} = \sum_{l_1, l_2, i_0, \dots, i_n \geq 0} (x - x_0)^{l_1} (t - t_0)^{l_2} (u_0 - a_0)^{i_0} \dots (u_n - a_n)^{i_n} \cdot \tilde{A}_{i_0 \dots i_n}^{l_1, l_2},$$

$$(44) \quad \tilde{B} = \sum_{l_1, l_2, j_0, \dots, j_{n+d-1} \geq 0} (x - x_0)^{l_1} (t - t_0)^{l_2} (u_0 - a_0)^{j_0} \dots (u_{n+d-1} - a_{n+d-1})^{j_{n+d-1}} \cdot \tilde{B}_{j_0 \dots j_{n+d-1}}^{l_1, l_2},$$

$$\tilde{A}_{i_0 \dots i_n}^{l_1, l_2}, \tilde{B}_{j_0 \dots j_{n+d-1}}^{l_1, l_2} \in \mathfrak{g}.$$

For each $n \in \mathbb{Z}_{\geq 0}$, set

$$(45) \quad \mathcal{V}_n = \left\{ (i_0, \dots, i_n) \in \mathbb{Z}_{\geq 0}^n \mid \exists r \in \{1, \dots, n\} \text{ such that } i_r = 1, i_q = 0 \ \forall q > r \right\}.$$

In other words, for $i_0, \dots, i_n \in \mathbb{Z}_{\geq 0}$, one has $(i_0, \dots, i_n) \in \mathcal{V}_n$ iff there is $r \in \{1, \dots, n\}$ such that $(i_0, \dots, i_{r-1}, i_r, i_{r+1}, \dots, i_n) = (i_0, \dots, i_{r-1}, 1, 0, \dots, 0)$. In particular, for $n = 0$ we have $\mathcal{V}_0 = \emptyset$.

Using formulas (43), (44), we see that properties (32), (33), (34) are equivalent to

$$(46) \quad \tilde{A}_{0\dots 0}^{l_1, l_2} = \tilde{B}_{0\dots 0}^{l_1, l_2} = \tilde{A}_{i_0\dots i_n}^{l_1, l_2} = 0, \quad (i_0, \dots, i_n) \in \mathcal{V}_n, \quad l_1, l_2 \in \mathbb{Z}_{\geq 0}.$$

Remark 11. Let \mathfrak{L} be a Lie algebra. Consider a formal power series of the form

$$C = \sum_{l_1, l_2, i_0, \dots, i_m \geq 0} (x - x_0)^{l_1} (t - t_0)^{l_2} (u_0 - a_0)^{i_0} \dots (u_m - a_m)^{i_m} \cdot C_{i_0\dots i_m}^{l_1, l_2}, \quad C_{i_0\dots i_m}^{l_1, l_2} \in \mathfrak{L}.$$

Set

$$(47) \quad D_x(C) = \sum_{l_1, l_2, i_0, \dots, i_m} D_x((x - x_0)^{l_1} (t - t_0)^{l_2} (u_0 - a_0)^{i_0} \dots (u_m - a_m)^{i_m}) \cdot C_{i_0\dots i_m}^{l_1, l_2},$$

$$(48) \quad D_t(C) = \sum_{l_1, l_2, i_0, \dots, i_m} D_t((x - x_0)^{l_1} (t - t_0)^{l_2} (u_0 - a_0)^{i_0} \dots (u_m - a_m)^{i_m}) \cdot C_{i_0\dots i_m}^{l_1, l_2}.$$

The expressions

$$(49) \quad \begin{aligned} & D_x((x - x_0)^{l_1} (t - t_0)^{l_2} (u_0 - a_0)^{i_0} \dots (u_m - a_m)^{i_m}), \\ & D_t((x - x_0)^{l_1} (t - t_0)^{l_2} (u_0 - a_0)^{i_0} \dots (u_m - a_m)^{i_m}) \end{aligned}$$

are functions of the variables x, t, u_k . Taking the corresponding Taylor series at the point (28), we regard (49) as power series. Then (47), (48) become formal power series with coefficients in \mathfrak{L} .

Consider another formal power series

$$R = \sum_{q_1, q_2, j_0, \dots, j_m \geq 0} (x - x_0)^{q_1} (t - t_0)^{q_2} (u_0 - a_0)^{j_0} \dots (u_m - a_m)^{j_m} \cdot R_{j_0\dots j_m}^{q_1, q_2}, \quad R_{j_0\dots j_m}^{q_1, q_2} \in \mathfrak{L}.$$

Then the Lie bracket $[C, R]$ is defined as follows

$$[C, R] = \sum_{\substack{l_1, l_2, i_0, \dots, i_m, \\ q_1, q_2, j_0, \dots, j_m}} (x - x_0)^{l_1 + q_1} (t - t_0)^{l_2 + q_2} (u_0 - a_0)^{i_0 + j_0} \dots (u_m - a_m)^{i_m + j_m} \cdot [C_{i_0\dots i_m}^{l_1, l_2}, R_{j_0\dots j_m}^{q_1, q_2}].$$

Remark 12. The main idea of the definition of the Lie algebra $\mathbb{F}^n(\mathcal{E}, a)$ can be informally outlined as follows. According to Theorem 6 and Remark 10, any ZCR (29), (30) of order $\leq n$ is gauge equivalent to a ZCR given by functions \tilde{A}, \tilde{B} that are of the form (43), (44) and satisfy (42), (46).

To define $\mathbb{F}^n(\mathcal{E}, a)$, we regard $\tilde{A}_{i_0\dots i_n}^{l_1, l_2}, \tilde{B}_{j_0\dots j_{n+d-1}}^{l_1, l_2}$ from (43), (44) as abstract symbols. By definition, the algebra $\mathbb{F}^n(\mathcal{E}, a)$ is generated by the symbols $\tilde{A}_{i_0\dots i_n}^{l_1, l_2}, \tilde{B}_{j_0\dots j_{n+d-1}}^{l_1, l_2}$ for $l_1, l_2, i_0, \dots, i_n, j_0, \dots, j_{n+d-1} \in \mathbb{Z}_{\geq 0}$. Relations for these generators are provided by equations (42), (46). The details of this construction are presented below.

Let \mathfrak{F} be the free Lie algebra generated by the symbols $\mathbf{A}_{i_0\dots i_n}^{l_1, l_2}, \mathbf{B}_{j_0\dots j_{n+d-1}}^{l_1, l_2}$ for $l_1, l_2, i_0, \dots, i_n, j_0, \dots, j_{n+d-1} \in \mathbb{Z}_{\geq 0}$. Consider the following power series with coefficients in \mathfrak{F}

$$\begin{aligned} \mathbf{A} &= \sum_{l_1, l_2, i_0, \dots, i_n \geq 0} (x - x_0)^{l_1} (t - t_0)^{l_2} (u_0 - a_0)^{i_0} \dots (u_n - a_n)^{i_n} \cdot \mathbf{A}_{i_0\dots i_n}^{l_1, l_2}, \\ \mathbf{B} &= \sum_{l_1, l_2, j_0, \dots, j_{n+d-1} \geq 0} (x - x_0)^{l_1} (t - t_0)^{l_2} (u_0 - a_0)^{j_0} \dots (u_{n+d-1} - a_{n+d-1})^{j_{n+d-1}} \cdot \mathbf{B}_{j_0\dots j_{n+d-1}}^{l_1, l_2}. \end{aligned}$$

Then the power series $D_x(\mathbf{B})$, $D_t(\mathbf{A})$, $[\mathbf{A}, \mathbf{B}]$ are defined according to Remark 11. We have

$$D_x(\mathbf{B}) - D_t(\mathbf{A}) + [\mathbf{A}, \mathbf{B}] = \sum_{l_1, l_2, q_0, \dots, q_{n+d} \geq 0} (x - x_0)^{l_1} (t - t_0)^{l_2} (u_0 - a_0)^{q_0} \dots (u_{n+d} - a_{n+d})^{q_{n+d}} \cdot \mathbf{Z}_{q_0 \dots q_{n+d}}^{l_1, l_2}$$

for some elements $\mathbf{Z}_{q_0 \dots q_{n+d}}^{l_1, l_2} \in \mathfrak{F}$.

Let $\mathfrak{I} \subset \mathfrak{F}$ be the ideal generated by the elements

$$\begin{aligned} \mathbf{Z}_{q_0 \dots q_{n+d}}^{l_1, l_2}, \quad \mathbf{A}_{0 \dots 0}^{l_1, l_2}, \quad \mathbf{B}_{0 \dots 0}^{0, l_2}, \quad l_1, l_2, q_0, \dots, q_{n+d} \in \mathbb{Z}_{\geq 0}, \\ \mathbf{A}_{i_0 \dots i_n}^{l_1, l_2}, \quad (i_0, \dots, i_n) \in \mathcal{V}_n, \quad l_1, l_2 \in \mathbb{Z}_{\geq 0}. \end{aligned}$$

Set $\mathbb{F}^n(\mathcal{E}, a) = \mathfrak{F}/\mathfrak{I}$. Consider the natural homomorphism $\rho: \mathfrak{F} \rightarrow \mathfrak{F}/\mathfrak{I} = \mathbb{F}^n(\mathcal{E}, a)$ and set

$$\mathbb{A}_{i_0 \dots i_n}^{l_1, l_2} = \rho(\mathbf{A}_{i_0 \dots i_n}^{l_1, l_2}), \quad \mathbb{B}_{j_0 \dots j_{n+d-1}}^{l_1, l_2} = \rho(\mathbf{B}_{j_0 \dots j_{n+d-1}}^{l_1, l_2}).$$

The definition of \mathfrak{I} implies that the power series

$$(50) \quad \mathbb{A} = \sum_{l_1, l_2, i_0, \dots, i_n \geq 0} (x - x_0)^{l_1} (t - t_0)^{l_2} (u_0 - a_0)^{i_0} \dots (u_n - a_n)^{i_n} \cdot \mathbb{A}_{i_0 \dots i_n}^{l_1, l_2},$$

$$(51) \quad \mathbb{B} = \sum_{l_1, l_2, j_0, \dots, j_{n+d-1} \geq 0} (x - x_0)^{l_1} (t - t_0)^{l_2} (u_0 - a_0)^{j_0} \dots (u_{n+d-1} - a_{n+d-1})^{j_{n+d-1}} \cdot \mathbb{B}_{j_0 \dots j_{n+d-1}}^{l_1, l_2}$$

satisfy

$$(52) \quad D_x(\mathbb{B}) - D_t(\mathbb{A}) + [\mathbb{A}, \mathbb{B}] = 0.$$

Remark 13. The Lie algebra $\mathbb{F}^n(\mathcal{E}, a)$ can be described in terms of generators and relations as follows. Equation (52) is equivalent to some Lie algebraic relations for $\mathbb{A}_{i_0 \dots i_n}^{l_1, l_2}$, $\mathbb{B}_{j_0 \dots j_{n+d-1}}^{l_1, l_2}$. The algebra $\mathbb{F}^n(\mathcal{E}, a)$ is given by the generators $\mathbb{A}_{i_0 \dots i_n}^{l_1, l_2}$, $\mathbb{B}_{j_0 \dots j_{n+d-1}}^{l_1, l_2}$, the relations arising from (52), and the following relations

$$(53) \quad \mathbb{A}_{0 \dots 0}^{l_1, l_2} = \mathbb{B}_{0 \dots 0}^{0, l_2} = \mathbb{A}_{i_0 \dots i_n}^{l_1, l_2} = 0, \quad (i_0, \dots, i_n) \in \mathcal{V}_n, \quad l_1, l_2 \in \mathbb{Z}_{\geq 0}.$$

Let \mathfrak{g} be a finite-dimensional Lie algebra. A homomorphism $\rho: \mathbb{F}^n(\mathcal{E}, a) \rightarrow \mathfrak{g}$ is said to be *tame* if the power series

$$(54) \quad \tilde{A} = \sum_{l_1, l_2, i_0, \dots, i_n} (x - x_0)^{l_1} (t - t_0)^{l_2} (u_0 - a_0)^{i_0} \dots (u_n - a_n)^{i_n} \cdot \rho(\mathbb{A}_{i_0 \dots i_n}^{l_1, l_2}),$$

$$(55) \quad \tilde{B} = \sum_{l_1, l_2, j_0, \dots, j_{n+d-1}} (x - x_0)^{l_1} (t - t_0)^{l_2} (u_0 - a_0)^{j_0} \dots (u_{n+d-1} - a_{n+d-1})^{j_{n+d-1}} \cdot \rho(\mathbb{B}_{j_0 \dots j_{n+d-1}}^{l_1, l_2})$$

are absolutely convergent on a neighborhood of $a \in \mathcal{E}$. In other words, ρ is tame iff (54), (55) are analytic functions with values in \mathfrak{g} on a neighborhood of $a \in \mathcal{E}$.

Since (50), (51) obey (52), the power series (54), (55) satisfy (42) for any homomorphism $\rho: \mathbb{F}^n(\mathcal{E}, a) \rightarrow \mathfrak{g}$. Therefore, if ρ is tame, the analytic functions (54), (55) form a ZCR. Denote this ZCR by $\mathbf{Z}(\mathcal{E}, a, n, \rho)$.

Combining this construction with Theorem 6 and Remark 10, we obtain the following result.

Theorem 7. *Let \mathfrak{g} be a finite-dimensional matrix Lie algebra. For any \mathfrak{g} -valued ZCR (29), (30) of order $\leq n$ on a neighborhood of $a \in \mathcal{E}$, there is a tame homomorphism $\rho: \mathbb{F}^n(\mathcal{E}, a) \rightarrow \mathfrak{g}$ such that the ZCR (29), (30) is gauge equivalent to the ZCR $\mathbf{Z}(\mathcal{E}, a, n, \rho)$.*

The ZCR $\mathbf{Z}(\mathcal{E}, a, n, \rho)$ takes values in the Lie algebra $\rho(\mathbb{F}^n(\mathcal{E}, a)) \subset \mathfrak{g}$.

Suppose that $n \geq 1$. According to Remark 13, the algebra $\mathbb{F}^n(\mathcal{E}, a)$ is given by the generators $\mathbb{A}_{i_0 \dots i_n}^{l_1, l_2}$, $\mathbb{B}_{j_0 \dots j_{n+d-1}}^{l_1, l_2}$ and the relations arising from (52), (53).

Similarly, the algebra $\mathbb{F}^{n-1}(\mathcal{E}, a)$ is given by the generators $\hat{\mathbb{A}}_{i_0 \dots i_{n-1}}^{l_1, l_2}, \hat{\mathbb{B}}_{j_0 \dots j_{n+d-2}}^{l_1, l_2}$ and the relations arising from

$$D_x(\hat{\mathbb{B}}) - D_t(\hat{\mathbb{A}}) + [\hat{\mathbb{A}}, \hat{\mathbb{B}}] = 0,$$

$$\hat{\mathbb{A}}_{0 \dots 0}^{l_1, l_2} = \hat{\mathbb{B}}_{0 \dots 0}^{l_1, l_2} = \hat{\mathbb{A}}_{i_0 \dots i_{n-1}}^{l_1, l_2} = 0, \quad (i_0, \dots, i_{n-1}) \in \mathcal{V}_{n-1}, \quad l_1, l_2 \in \mathbb{Z}_{\geq 0},$$

where

$$\hat{\mathbb{A}} = \sum_{l_1, l_2, i_0, \dots, i_{n-1}} (x - x_0)^{l_1} (t - t_0)^{l_2} (u_0 - a_0)^{i_0} \dots (u_{n-1} - a_{n-1})^{i_{n-1}} \cdot \hat{\mathbb{A}}_{i_0 \dots i_{n-1}}^{l_1, l_2},$$

$$\hat{\mathbb{B}} = \sum_{l_1, l_2, j_0, \dots, j_{n+d-2}} (x - x_0)^{l_1} (t - t_0)^{l_2} (u_0 - a_0)^{j_0} \dots (u_{n+d-2} - a_{n+d-2})^{j_{n+d-2}} \cdot \hat{\mathbb{B}}_{j_0 \dots j_{n+d-2}}^{l_1, l_2}.$$

This implies that the map

$$\hat{\mathbb{A}}_{i_0 \dots i_{n-1}}^{l_1, l_2} \mapsto \delta_{0, i_n} \cdot \hat{\mathbb{A}}_{i_0 \dots i_{n-1}}^{l_1, l_2}, \quad \hat{\mathbb{B}}_{j_0 \dots j_{n+d-2}}^{l_1, l_2} \mapsto \delta_{0, j_{n+d-1}} \cdot \hat{\mathbb{B}}_{j_0 \dots j_{n+d-2}}^{l_1, l_2}$$

determines a surjective homomorphism $\mathbb{F}^n(\mathcal{E}, a) \rightarrow \mathbb{F}^{n-1}(\mathcal{E}, a)$. Here δ_{0, i_n} and $\delta_{0, j_{n+d-1}}$ are the Kronecker deltas.

According to Theorem 7, the algebra $\mathbb{F}^n(\mathcal{E}, a)$ is responsible for ZCRs of order $\leq n$, and the algebra $\mathbb{F}^{n-1}(\mathcal{E}, a)$ is responsible for ZCRs of order $\leq n-1$. The constructed homomorphism $\mathbb{F}^n(\mathcal{E}, a) \rightarrow \mathbb{F}^{n-1}(\mathcal{E}, a)$ reflects the fact that any ZCR of order $\leq n-1$ is at the same time of order $\leq n$.

Thus we obtain the following sequence of surjective homomorphisms of Lie algebras

$$(56) \quad \dots \rightarrow \mathbb{F}^n(\mathcal{E}, a) \rightarrow \mathbb{F}^{n-1}(\mathcal{E}, a) \rightarrow \dots \rightarrow \mathbb{F}^1(\mathcal{E}, a) \rightarrow \mathbb{F}^0(\mathcal{E}, a).$$

3. SOME RESULTS ON GENERATORS OF $\mathbb{F}^n(\mathcal{E}, a)$

According to Remark 13, the algebra $\mathbb{F}^n(\mathcal{E}, a)$ is given by the generators $\mathbb{A}_{i_0 \dots i_n}^{l_1, l_2}, \mathbb{B}_{j_0 \dots j_{n+d-1}}^{l_1, l_2}$ and the relations arising from (52), (53). Using (4), we can rewrite equation (52) as

$$(57) \quad \frac{\partial}{\partial x}(\mathbb{B}) + \sum_{k=0}^{n+d-1} u_{k+1} \frac{\partial}{\partial u_k}(\mathbb{B}) - \frac{\partial}{\partial t}(\mathbb{A}) - \sum_{k=0}^n D_x^k(F(x, t, u_0, u_1, \dots, u_d)) \frac{\partial}{\partial u_k}(\mathbb{A}) + [\mathbb{A}, \mathbb{B}] = 0.$$

In this section, we regard $F = F(x, t, u_0, u_1, \dots, u_d)$ as a power series, using the Taylor series of the function F at the point (28).

Proposition 5. *The elements*

$$(58) \quad \mathbb{A}_{i_0 \dots i_n}^{l_1, 0}, \quad l_1, i_0, \dots, i_n \in \mathbb{Z}_{\geq 0},$$

generate the algebra $\mathbb{F}^n(\mathcal{E}, a)$.

Proof. For each $l \in \mathbb{Z}_{\geq 0}$, denote by $\mathfrak{f}_l \subset \mathbb{F}^n(\mathcal{E}, a)$ the subalgebra generated by all the elements $\mathbb{A}_{i_0 \dots i_n}^{l_1, l_2}$ with $l_2 \leq l$.

Lemma 1. *Let $l_1, l_2, j_0, \dots, j_{n+d-1} \in \mathbb{Z}_{\geq 0}$ be such that $j_0 + \dots + j_{n+d-1} > 0$. Then $\mathbb{B}_{j_0 \dots j_{n+d-1}}^{l_1, l_2} \in \mathfrak{f}_{l_2}$.*

Proof. For any $j_0, \dots, j_{n+d-1} \in \mathbb{Z}_{\geq 0}$ satisfying $j_0 + \dots + j_{n+d-1} > 0$, denote by $\rho(j_0, \dots, j_{n+d-1})$ the maximal integer $r \in \{0, 1, \dots, n+d-1\}$ such that $j_r \neq 0$. Set also $\rho(0, \dots, 0) = -1$.

Differentiating (57) with respect to u_{n+d} , we obtain

$$\frac{\partial}{\partial u_{n+d-1}}(\mathbb{B}) = \frac{\partial F}{\partial u_d} \cdot \frac{\partial}{\partial u_n}(\mathbb{A}),$$

which implies $\mathbb{B}_{j_0 \dots j_{n+d-1}}^{l_1, l_2} \in \mathfrak{f}_{l_2}$ for all $l_1, l_2, j_0, \dots, j_{n+d-1} \in \mathbb{Z}_{\geq 0}$ obeying $\rho(j_0, \dots, j_{n+d-1}) = n+d-1$.

Let $m \in \{0, 1, \dots, n + d - 1\}$ be such that

$$(59) \quad \mathbb{B}_{j'_0 \dots j'_{n+d-1}}^{l_1, l_2} \in \mathfrak{f}_{l_2} \quad \text{for all } l_1, l_2, j'_0, \dots, j'_{n+d-1} \in \mathbb{Z}_{\geq 0} \text{ satisfying } \rho(j'_0, \dots, j'_{n+d-1}) > m.$$

We are going to show that $\mathbb{B}_{\tilde{j}_0 \dots \tilde{j}_{n+d-1}}^{l_1, l_2} \in \mathfrak{f}_{l_2}$ for all $l_1, l_2, \tilde{j}_0, \dots, \tilde{j}_{n+d-1} \in \mathbb{Z}_{\geq 0}$ satisfying $\rho(\tilde{j}_0, \dots, \tilde{j}_{n+d-1}) = m$.

For any power series C of the form

$$C = \sum_{l_1, l_2, d_0, \dots, d_k \geq 0} (x - x_0)^{l_1} (t - t_0)^{l_2} (u_0 - a_0)^{d_0} \dots (u_k - a_k)^{d_k} \cdot C_{d_0 \dots d_k}^{l_1, l_2}, \quad C_{d_0 \dots d_k}^{l_1, l_2} \in \mathbb{F}^n(\mathcal{E}, a),$$

set

$$\mathbf{S}(C) = \left(\frac{\partial}{\partial u_{m+1}}(C) \right) \Big|_{u_k = a_k, k \geq m+1}.$$

That is, in order to obtain $\mathbf{S}(C)$, we differentiate C with respect to u_{m+1} and then substitute $u_k = a_k$ for all $k \geq m + 1$.

Property (53) implies

$$(60) \quad \mathbf{S}\left(\frac{\partial}{\partial t}(\mathbb{A})\right) = 0.$$

Combining (57) with (60), we get

$$(61) \quad \mathbf{S}(D_x(\mathbb{B})) = \mathbf{S}\left(\sum_{k=0}^n D_x^k(F) \frac{\partial}{\partial u_k}(\mathbb{A})\right) - \mathbf{S}([\mathbb{A}, \mathbb{B}]).$$

Using (51), one obtains

$$(62) \quad \mathbf{S}(D_x(\mathbb{B})) = \sum_{\substack{l_1, l_2, j_0, \dots, j_{n+d-1} \geq 0, \\ \rho(j_0, \dots, j_{n+d-1}) = m}} j_m (x - x_0)^{l_1} (t - t_0)^{l_2} (u_0 - a_0)^{j_0} \dots (u_m - a_m)^{j_m-1} \mathbb{B}_{j_0 \dots j_{n+d-1}}^{l_1, l_2} + \\ + \mathbf{S}\left(\sum_{\substack{l_1, l_2, j_0, \dots, j_{n+d-1} \geq 0, \\ \rho(j_0, \dots, j_{n+d-1}) > m}} (t - t_0)^{l_2} D_x\left((x - x_0)^{l_1} (u_0 - a_0)^{j_0} \dots (u_{n+d-1} - a_{n+d-1})^{j_{n+d-1}}\right) \cdot \mathbb{B}_{j_0 \dots j_{n+d-1}}^{l_1, l_2}\right).$$

From (53) it follows that $\mathbf{S}(\mathbb{A}) = 0$, which yields

$$(63) \quad \mathbf{S}([\mathbb{A}, \mathbb{B}]) = \left[\mathbf{S}(\mathbb{A}), \mathbb{B} \Big|_{u_k = a_k, k \geq m+1} \right] + \left[\mathbb{A} \Big|_{u_k = a_k, k \geq m+1}, \mathbf{S}(\mathbb{B}) \right] = \left[\mathbb{A} \Big|_{u_k = a_k, k \geq m+1}, \mathbf{S}(\mathbb{B}) \right].$$

In view of (62), (63), for any $l_1, l_2, \tilde{j}_0, \dots, \tilde{j}_{n+d-1} \in \mathbb{Z}_{\geq 0}$ satisfying $\rho(\tilde{j}_0, \dots, \tilde{j}_{n+d-1}) = m$ the element $\mathbb{B}_{\tilde{j}_0 \dots \tilde{j}_{n+d-1}}^{l_1, l_2}$ appears only once on the left-hand side of (61) and does not appear on the right-hand side of (61).

Combining (61), (62), (63), we see that the element $\mathbb{B}_{\tilde{j}_0 \dots \tilde{j}_{n+d-1}}^{l_1, l_2}$ is equal to a linear combination of elements of the form

$$(64) \quad \mathbb{A}_{i_0 \dots i_n}^{l'_1, l'_2}, \quad \mathbb{B}_{\hat{j}_0 \dots \hat{j}_{n+d-1}}^{\hat{l}_1, \hat{l}_2}, \quad \left[\mathbb{A}_{i_0 \dots i_n}^{l'_1, l'_2}, \mathbb{B}_{\hat{j}_0 \dots \hat{j}_{n+d-1}}^{\hat{l}_1, \hat{l}_2} \right], \quad l'_2 \leq l_2, \quad \hat{l}_2 \leq l_2, \quad \rho(\hat{j}_0, \dots, \hat{j}_{n+d-1}) > m.$$

Obviously, for any $\hat{l}_2 \leq l_2$ one has $\mathfrak{f}_{\hat{l}_2} \subset \mathfrak{f}_{l_2}$. Taking into account assumption (59), we obtain that the elements (64) belong to \mathfrak{f}_{l_2} . Hence $\mathbb{B}_{\tilde{j}_0 \dots \tilde{j}_{n+d-1}}^{l_1, l_2} \in \mathfrak{f}_{l_2}$.

The proof is completed by induction. \square

Lemma 2. For all $l_1, l_2 \in \mathbb{Z}_{\geq 0}$, one has $\mathbb{B}_{0 \dots 0}^{l_1, l_2} \in \mathfrak{f}_{l_2}$.

Proof. According to (53), we have $\mathbb{B}_{0\dots 0}^{0,l_2} = 0$. Therefore, it is sufficient to prove $\mathbb{B}_{0\dots 0}^{l_1,l_2} \in \mathfrak{f}_{l_2}$ for $l_1 > 0$.

Note that property (53) implies

$$(65) \quad \mathbb{A} \Big|_{u_k=a_k, k \geq 0} = 0, \quad \frac{\partial}{\partial t}(\mathbb{A}) \Big|_{u_k=a_k, k \geq 0} = 0.$$

In view of (51), one has

$$(66) \quad \frac{\partial}{\partial x}(\mathbb{B}) \Big|_{u_k=a_k, k \geq 0} = \sum_{l_1 > 0, l_2 \geq 0} l_1(x-x_0)^{l_1-1}(t-t_0)^{l_2} \cdot \mathbb{B}_{0\dots 0}^{l_1,l_2}.$$

Substituting $u_k = a_k$ for all $k \in \mathbb{Z}_{\geq 0}$ in (57) and using (65), (66), we get

$$(67) \quad \sum_{l_1 > 0, l_2 \geq 0} l_1(x-x_0)^{l_1-1}(t-t_0)^{l_2} \cdot \mathbb{B}_{0\dots 0}^{l_1,l_2} = \\ = - \left(\sum_{k=0}^{n+d-1} u_{k+1} \frac{\partial}{\partial u_k}(\mathbb{B}) \right) \Big|_{u_k=a_k, k \geq 0} + \left(\sum_{k=0}^n D_x^k(F) \frac{\partial}{\partial u_k}(\mathbb{A}) \right) \Big|_{u_k=a_k, k \geq 0}.$$

Combining (50), (51), (67), we see that for any $l_1 > 0$ and $l_2 \geq 0$ the element $\mathbb{B}_{0\dots 0}^{l_1,l_2}$ is equal to a linear combination of elements of the form

$$(68) \quad \mathbb{A}_{i_0\dots i_n}^{l'_1,l_2}, \quad \mathbb{B}_{j_0\dots j_{n+d-1}}^{l'_1,l_2}, \quad l'_1, i_0, \dots, i_n, j_0, \dots, j_{n+d-1} \in \mathbb{Z}_{\geq 0}, \quad j_0 + \dots + j_{n+d-1} = 1.$$

According to Lemma 1 and the definition of \mathfrak{f}_{l_2} , the elements (68) belong to \mathfrak{f}_{l_2} . Thus $\mathbb{B}_{0\dots 0}^{l_1,l_2} \in \mathfrak{f}_{l_2}$. \square

Lemma 3. For all $l_1, l, i_0, \dots, i_n \in \mathbb{Z}_{\geq 0}$, we have $\mathbb{A}_{i_0\dots i_n}^{l_1,l+1} \in \mathfrak{f}_l$.

Proof. Using (50), we can rewrite equation (57) as

$$\sum_{l_1, l, i_0, \dots, i_n \geq 0} (l+1)(x-x_0)^{l_1}(t-t_0)^l (u_0-a_0)^{i_0} \dots (u_n-a_n)^{i_n} \cdot \mathbb{A}_{i_0\dots i_n}^{l_1,l+1} = \\ = \frac{\partial}{\partial x}(\mathbb{B}) + \sum_{k=0}^{n+d-1} u_{k+1} \frac{\partial}{\partial u_k}(\mathbb{B}) - \sum_{k=0}^n D_x^k(F) \frac{\partial}{\partial u_k}(\mathbb{A}) + [\mathbb{A}, \mathbb{B}].$$

This implies that $\mathbb{A}_{i_0\dots i_n}^{l_1,l+1}$ is equal to a linear combination of elements of the form

$$(69) \quad \mathbb{A}_{i_0\dots i_n}^{\hat{l}_1, \hat{l}_2}, \quad \mathbb{B}_{j_0\dots j_{n+d-1}}^{\tilde{l}_1, \tilde{l}_2}, \quad \left[\mathbb{A}_{i_0\dots i_n}^{\hat{l}_1, \hat{l}_2}, \mathbb{B}_{j_0\dots j_{n+d-1}}^{\tilde{l}_1, \tilde{l}_2} \right], \quad \hat{l}_2 \leq l, \quad \tilde{l}_2 \leq l, \quad i_0, \dots, i_n, j_0, \dots, j_{n+d-1} \in \mathbb{Z}_{\geq 0}.$$

Using Lemmas 1, 2 and the condition $\tilde{l}_2 \leq l$, we get $\mathbb{B}_{j_0\dots j_{n+d-1}}^{\tilde{l}_1, \tilde{l}_2} \in \mathfrak{f}_{\tilde{l}_2} \subset \mathfrak{f}_l$. Therefore, the elements (69) belong to \mathfrak{f}_l . Hence $\mathbb{A}_{i_0\dots i_n}^{l_1,l+1} \in \mathfrak{f}_l$. \square

Return to the proof of Proposition 5. According to Lemmas 1, 2 and the definition of \mathfrak{f}_l , we have $\mathbb{A}_{i_0\dots i_n}^{l_1,l_2}, \mathbb{B}_{j_0\dots j_{n+d-1}}^{l_1,l_2} \in \mathfrak{f}_{l_2}$ for all $l_1, l_2, i_0, \dots, i_n, j_0, \dots, j_{n+d-1} \in \mathbb{Z}_{\geq 0}$. Lemma 3 implies that

$$\mathfrak{f}_{l_2} \subset \mathfrak{f}_{l_2-1} \subset \mathfrak{f}_{l_2-2} \subset \dots \subset \mathfrak{f}_0.$$

Therefore, $\mathbb{F}^n(\mathcal{E}, a)$ is equal to \mathfrak{f}_0 , which is generated by the elements (58). \square

4. THE HOMOMORPHISMS $\mathbb{F}^n(\mathcal{E}, a) \rightarrow \mathbb{F}^{n-1}(\mathcal{E}, a)$ AND $\mathbb{F}^n(\mathcal{E}, a) \rightarrow \mathbb{F}^0(\mathcal{E}, a)$ FOR EQUATIONS (15)

In this section we study the algebras (56) for equations of the form

$$(70) \quad u_t = u_{2q+1} + f(x, t, u_0, u_1, \dots, u_{2q-1}), \quad q \in \mathbb{Z}_{>0},$$

where f is an arbitrary function.

Let \mathcal{E} be the infinite prolongation of equation (70). Then \mathcal{E} is an infinite-dimensional manifold with the coordinates x, t, u_k for $k \in \mathbb{Z}_{\geq 0}$.

For equation (70), the total derivative operators (4) are

$$(71) \quad D_x = \frac{\partial}{\partial x} + \sum_{k \geq 0} u_{k+1} \frac{\partial}{\partial u_k}, \quad D_t = \frac{\partial}{\partial t} + \sum_{k \geq 0} D_x^k (u_{2q+1} + f(x, t, u_0, u_1, \dots, u_{2q-1})) \frac{\partial}{\partial u_k}.$$

Consider an arbitrary point $a \in \mathcal{E}$ given by

$$(72) \quad a = (x = x_0, t = t_0, u_k = a_k) \in \mathcal{E}, \quad x_0, t_0, a_k \in \mathbb{K}, \quad k \in \mathbb{Z}_{\geq 0}.$$

Let $n \in \mathbb{Z}_{>0}$ be such that $n \geq 2q - 1$. According to Remark 13, the algebra $\mathbb{F}^n(\mathcal{E}, a)$ can be described as follows. Consider formal power series

$$(73) \quad \mathbb{A} = \sum_{l_1, l_2, i_0, \dots, i_n \geq 0} (x - x_0)^{l_1} (t - t_0)^{l_2} (u_0 - a_0)^{i_0} \dots (u_n - a_n)^{i_n} \cdot \mathbb{A}_{i_0 \dots i_n}^{l_1, l_2},$$

$$(74) \quad \mathbb{B} = \sum_{l_1, l_2, j_0, \dots, j_{n+2q} \geq 0} (x - x_0)^{l_1} (t - t_0)^{l_2} (u_0 - a_0)^{j_0} \dots (u_{n+2q} - a_{n+2q})^{j_{n+2q}} \cdot \mathbb{B}_{j_0 \dots j_{n+2q}}^{l_1, l_2}$$

satisfying

$$(75) \quad \mathbb{A}_{i_0 \dots i_n}^{l_1, l_2} = 0 \quad \text{if} \quad \exists r \in \{1, \dots, n\} \quad \text{such that} \quad i_r = 1, \quad i_m = 0 \quad \forall m > r,$$

$$(76) \quad \mathbb{A}_{0 \dots 0}^{l_1, l_2} = 0 \quad \forall l_1, l_2 \in \mathbb{Z}_{\geq 0},$$

$$(77) \quad \mathbb{B}_{0 \dots 0}^{0, l_2} = 0 \quad \forall l_2 \in \mathbb{Z}_{\geq 0}.$$

Then $\mathbb{A}_{i_0 \dots i_n}^{l_1, l_2}, \mathbb{B}_{j_0 \dots j_{n+2q}}^{l_1, l_2}$ are generators of the algebra $\mathbb{F}^n(\mathcal{E}, a)$, and the equation

$$(78) \quad D_x(\mathbb{B}) - D_t(\mathbb{A}) + [\mathbb{A}, \mathbb{B}] = 0$$

provides relations for these generators (in addition to relations (75), (76), (77)).

Note that condition (75) is equivalent to

$$(79) \quad \left. \frac{\partial}{\partial u_s}(\mathbb{A}) \right|_{u_k = a_k, k \geq s} = 0 \quad \forall s \geq 1.$$

Using (71), we can rewrite equation (78) as

$$(80) \quad \begin{aligned} \frac{\partial}{\partial x}(\mathbb{B}) + \sum_{k=0}^{n+2q} u_{k+1} \frac{\partial}{\partial u_k}(\mathbb{B}) + [\mathbb{A}, \mathbb{B}] &= \\ &= \frac{\partial}{\partial t}(\mathbb{A}) + \sum_{k=0}^n \left(u_{k+2q+1} + D_x^k (f(x, t, u_0, u_1, \dots, u_{2q-1})) \right) \frac{\partial}{\partial u_k}(\mathbb{A}). \end{aligned}$$

Here we regard $f(x, t, u_0, u_1, \dots, u_{2q-1})$ as a power series, using the Taylor series of the function f at the point (72).

Differentiating (80) with respect to u_{n+2q+1} , we obtain

$$(81) \quad \frac{\partial}{\partial u_{n+2q}}(\mathbb{B}) = \frac{\partial}{\partial u_n}(\mathbb{A}).$$

From (81) it follows that \mathbb{B} is of the form

$$(82) \quad \mathbb{B} = u_{n+2q} \frac{\partial}{\partial u_n}(\mathbb{A}) + \mathbb{B}_0(x, t, u_0, \dots, u_{n+2q-1}),$$

where $\mathbb{B}_0(x, t, u_0, \dots, u_{n+2q-1})$ is a power series in the variables

$$x - x_0, \quad t - t_0, \quad u_0 - a_0, \quad \dots, \quad u_{n+2q-1} - a_{n+2q-1}.$$

Differentiating (80) with respect to u_{n+2q} , u_{n+i} for $i = 1, \dots, 2q - 1$ and using (82), one gets

$$\frac{\partial^2}{\partial u_n \partial u_n}(\mathbb{A}) + \frac{\partial^2}{\partial u_{n+1} \partial u_{n+2q-1}}(\mathbb{B}_0) = 0, \quad \frac{\partial^2}{\partial u_{n+s} \partial u_{n+2q-1}}(\mathbb{B}_0) = 0, \quad s = 2, \dots, 2q - 1.$$

Therefore, $\mathbb{B}_0 = \mathbb{B}_0(x, t, u_0, \dots, u_{n+2q-1})$ is of the form

$$(83) \quad \mathbb{B}_0 = u_{n+1} u_{n+2q-1} \left(\frac{1}{2} \delta_{q,1} - 1 \right) \frac{\partial^2}{\partial u_n \partial u_n}(\mathbb{A}) + \\ + u_{n+2q-1} \mathbb{B}_{01}(x, t, u_0, \dots, u_n) + \mathbb{B}_{00}(x, t, u_0, \dots, u_{n+2q-2}).$$

Here $\mathbb{B}_{01}(x, t, u_0, \dots, u_n)$ is a power series in the variables $x - x_0, t - t_0, u_0 - a_0, \dots, u_n - a_n$ and $\mathbb{B}_{00}(x, t, u_0, \dots, u_{n+2q-2})$ is a power series in the variables $x - x_0, t - t_0, u_0 - a_0, \dots, u_{n+2q-2} - a_{n+2q-2}$.

Recall that $n \geq 2q - 1 \geq 1$. Applying the operator $\frac{\partial^3}{\partial u_{n+1} \partial u_{n+1} \partial u_{n+2q-1}}$ to equation (80) and using (82), (83), we get

$$\frac{\partial^3}{\partial u_n \partial u_n \partial u_n}(\mathbb{A}) = 0.$$

Hence \mathbb{A} is of the form

$$(84) \quad \mathbb{A} = (u_n - a_n)^2 \mathbb{A}_2(x, t, u_0, \dots, u_{n-1}) + (u_n - a_n) \mathbb{A}_1(x, t, u_0, \dots, u_{n-1}) + \mathbb{A}_0(x, t, u_0, \dots, u_{n-1}),$$

where $\mathbb{A}_j(x, t, u_0, \dots, u_{n-1})$ is a power series in the variables $x - x_0, t - t_0, u_0 - a_0, \dots, u_{n-1} - a_{n-1}$ for $j = 0, 1, 2$.

Equation (79) for $s = n$ yields

$$(85) \quad \mathbb{A}_1(x, t, u_0, \dots, u_{n-1}) = 0.$$

Combining (82), (83), (84), (85), we get

$$(86) \quad \mathbb{B} = 2u_{n+2q}(u_n - a_n) \mathbb{A}_2(x, t, u_0, \dots, u_{n-1}) + u_{n+1} u_{n+2q-1} (\delta_{q,1} - 2) \mathbb{A}_2(x, t, u_0, \dots, u_{n-1}) + \\ + u_{n+2q-1} \mathbb{B}_{01}(x, t, u_0, \dots, u_n) + \mathbb{B}_{00}(x, t, u_0, \dots, u_{n+2q-2}).$$

Applying the operator $\frac{\partial^2}{\partial u_{n+1} \partial u_{n+2q-1}}$ to equation (80), one gets

$$(87) \quad -2D_x(\mathbb{A}_2) + (\delta_{q,1} + 1) \frac{\partial}{\partial u_n}(\mathbb{B}_{01}) - 2[\mathbb{A}_0, \mathbb{A}_2] = 0.$$

Differentiating (87) with respect to u_n , we obtain

$$(88) \quad -2 \frac{\partial}{\partial u_{n-1}}(\mathbb{A}_2) + (\delta_{q,1} + 1) \frac{\partial^2}{\partial u_n \partial u_n}(\mathbb{B}_{01}) = 0.$$

Applying the operator $\frac{\partial^3}{\partial u_n \partial u_n \partial u_{n+2q}}$ to equation (80), one gets

$$(89) \quad 4 \frac{\partial}{\partial u_{n-1}}(\mathbb{A}_2) + \frac{\partial^2}{\partial u_n \partial u_n}(\mathbb{B}_{01}) = 2 \frac{\partial}{\partial u_{n-1}}(\mathbb{A}_2).$$

Equations (88), (89) imply

$$(90) \quad \frac{\partial}{\partial u_{n-1}}(\mathbb{A}_2(x, t, u_0, \dots, u_{n-1})) = 0.$$

Applying the operator $\frac{\partial^2}{\partial u_n \partial u_{n+2q}}$ to equation (80) and using (90), we get

$$(91) \quad 2D_x(\mathbb{A}_2) + \frac{\partial}{\partial u_n}(\mathbb{B}_{01}) + 2[\mathbb{A}_0, \mathbb{A}_2] = 0.$$

Combining (91) with (87), we obtain

$$(92) \quad D_x(\mathbb{A}_2) + [\mathbb{A}_0, \mathbb{A}_2] = 0.$$

Lemma 4. *One has*

$$(93) \quad \frac{\partial}{\partial u_k}(\mathbb{A}_2) = 0 \quad \forall k \in \mathbb{Z}_{\geq 0}.$$

Proof. Suppose that (93) does not hold. Let k_0 be the maximal integer such that $\frac{\partial}{\partial u_{k_0}}(\mathbb{A}_2) \neq 0$.

From (90) it follows that $k_0 < n - 1$. Equation (79) for $s = k_0 + 1$ implies

$$(94) \quad \left. \frac{\partial}{\partial u_{k_0+1}}(\mathbb{A}_0) \right|_{u_k=a_k, k \geq k_0+1} = 0.$$

Differentiating (92) with respect to u_{k_0+1} , we obtain

$$(95) \quad \frac{\partial}{\partial u_{k_0}}(\mathbb{A}_2) + \left[\frac{\partial}{\partial u_{k_0+1}}(\mathbb{A}_0), \mathbb{A}_2 \right] = 0.$$

Substituting $u_k = a_k$ in (95) for all $k \geq k_0 + 1$ and using (94), one gets $\frac{\partial}{\partial u_{k_0}}(\mathbb{A}_2) = 0$, which contradicts to our assumption. \square

From (93) it follows that equation (92) reads

$$(96) \quad \frac{\partial}{\partial x}(\mathbb{A}_2) + [\mathbb{A}_0, \mathbb{A}_2] = 0.$$

Note that condition (76) implies

$$(97) \quad \left. \mathbb{A}_0 \right|_{u_k=a_k, k \geq 0} = 0.$$

Substituting $u_k = a_k$ in (96) for all $k \geq 0$ and using (93), (97), we get

$$(98) \quad \frac{\partial}{\partial x}(\mathbb{A}_2) = 0.$$

Combining (98) with (96), one obtains

$$(99) \quad [\mathbb{A}_2, \mathbb{A}_0] = 0.$$

In view of (73), (84), we have

$$(100) \quad \mathbb{A}_0 = \sum_{l_1, l_2, i_0, \dots, i_{n-1} \geq 0} (x - x_0)^{l_1} (t - t_0)^{l_2} (u_0 - a_0)^{i_0} \dots (u_{n-1} - a_{n-1})^{i_{n-1}} \cdot \mathbb{A}_{i_0 \dots i_{n-1} 0}^{l_1, l_2}$$

According to (73), (84), (93), (98), one has

$$(101) \quad \mathbb{A}_2 = \sum_{l \geq 0} (t - t_0)^l \cdot \tilde{\mathbb{A}}^l, \quad \tilde{\mathbb{A}}^l = \mathbb{A}_{0 \dots 0 2}^{0, l} \in \mathbb{F}^n(\mathcal{E}, a).$$

Combining (84), (85), (100), (101) with Proposition 5, we obtain that the elements

$$(102) \quad \tilde{\mathbb{A}}^0, \quad \mathbb{A}_{i_0 \dots i_{n-1} 0}^{l_1, 0}, \quad l_1, i_0, \dots, i_{n-1} \in \mathbb{Z}_{\geq 0},$$

generate the algebra $\mathbb{F}^n(\mathcal{E}, a)$.

Substituting $t = t_0$ in (99) and using (100), (101), one gets

$$(103) \quad [\tilde{\mathbb{A}}^0, \mathbb{A}_{i_0 \dots i_{n-1} 0}^{l_1, 0}] = 0 \quad \forall l_1, i_0, \dots, i_{n-1} \in \mathbb{Z}_{\geq 0}.$$

Since the elements (102) generate the algebra $\mathbb{F}^n(\mathcal{E}, a)$, equation (103) yields

$$(104) \quad [\tilde{\mathbb{A}}^0, \mathbb{F}^n(\mathcal{E}, a)] = 0.$$

Lemma 5. *One has*

$$(105) \quad [\tilde{\mathbb{A}}^l, \mathbb{F}^n(\mathcal{E}, a)] = 0 \quad \forall l \in \mathbb{Z}_{\geq 0}.$$

Proof. We prove (105) by induction on l . The property $[\tilde{\mathbb{A}}^0, \mathbb{F}^n(\mathcal{E}, a)] = 0$ was obtained in (104).

Let $r \in \mathbb{Z}_{\geq 0}$ be such that $[\tilde{\mathbb{A}}^l, \mathbb{F}^n(\mathcal{E}, a)] = 0$ for all $l \leq r$. Since $\left. \frac{\partial^l}{\partial t^l}(\mathbb{A}_2) \right|_{t=t_0} = l! \cdot \tilde{\mathbb{A}}^l$, we get

$$(106) \quad \left[\left. \frac{\partial^l}{\partial t^l}(\mathbb{A}_2) \right|_{t=t_0}, \left. \frac{\partial^m}{\partial t^m}(\mathbb{A}_0) \right|_{t=t_0} \right] = 0 \quad \forall l \leq r, \quad \forall m \in \mathbb{Z}_{\geq 0}.$$

Applying the operator $\frac{\partial^{r+1}}{\partial t^{r+1}}$ to equation (99), substituting $t = t_0$, and using (106), one obtains

$$\begin{aligned} 0 &= \frac{\partial^{r+1}}{\partial t^{r+1}}([\mathbb{A}_2, \mathbb{A}_0]) \Big|_{t=t_0} = \sum_{k=0}^{r+1} \binom{r+1}{k} \cdot \left[\left. \frac{\partial^k}{\partial t^k}(\mathbb{A}_2) \right|_{t=t_0}, \left. \frac{\partial^{r+1-k}}{\partial t^{r+1-k}}(\mathbb{A}_0) \right|_{t=t_0} \right] = \\ &= \left[\left. \frac{\partial^{r+1}}{\partial t^{r+1}}(\mathbb{A}_2) \right|_{t=t_0}, \mathbb{A}_0 \Big|_{t=t_0} \right] = \\ &= \left[(r+1)! \cdot \tilde{\mathbb{A}}^{r+1}, \sum_{l_1, i_0, \dots, i_{n-1}} (x - x_0)^{l_1} (u_0 - a_0)^{i_0} \dots (u_{n-1} - a_{n-1})^{i_{n-1}} \cdot \mathbb{A}_{i_0 \dots i_{n-1} 0}^{l_1, 0} \right], \end{aligned}$$

which implies

$$(107) \quad [\tilde{\mathbb{A}}^{r+1}, \mathbb{A}_{i_0 \dots i_{n-1} 0}^{l_1, 0}] = 0 \quad \forall l_1, i_0, \dots, i_{n-1} \in \mathbb{Z}_{\geq 0}.$$

Equation (104) yields

$$(108) \quad [\tilde{\mathbb{A}}^0, \tilde{\mathbb{A}}^{r+1}] = 0.$$

Since the elements (102) generate the algebra $\mathbb{F}^n(\mathcal{E}, a)$, from (107), (108) it follows that $[\tilde{\mathbb{A}}^{r+1}, \mathbb{F}^n(\mathcal{E}, a)] = 0$. \square

Theorem 8. *Let \mathcal{E} be the infinite prolongation of an equation of the form*

$$u_t = u_{2q+1} + f(x, t, u_0, u_1, \dots, u_{2q-1}), \quad q \in \mathbb{Z}_{>0}.$$

Let $a \in \mathcal{E}$. For each $n \in \mathbb{Z}_{>0}$, consider the homomorphism $\varphi_n: \mathbb{F}^n(\mathcal{E}, a) \rightarrow \mathbb{F}^{n-1}(\mathcal{E}, a)$ from (56).

If $n \geq 2q - 1$ then

$$(109) \quad [v_1, v_2] = 0 \quad \forall v_1 \in \ker \varphi_n, \quad \forall v_2 \in \mathbb{F}^n(\mathcal{E}, a).$$

In other words, if $n \geq 2q - 1$ then the kernel of φ_n is contained in the center of the Lie algebra $\mathbb{F}^n(\mathcal{E}, a)$.

For each $k \in \mathbb{Z}_{>0}$, let $\psi_k: \mathbb{F}^{k+2q-2}(\mathcal{E}, a) \rightarrow \mathbb{F}^{2q-2}(\mathcal{E}, a)$ be the composition of the homomorphisms

$$\mathbb{F}^{k+2q-2}(\mathcal{E}, a) \rightarrow \mathbb{F}^{k+2q-3}(\mathcal{E}, a) \rightarrow \dots \rightarrow \mathbb{F}^{2q-1}(\mathcal{E}, a) \rightarrow \mathbb{F}^{2q-2}(\mathcal{E}, a)$$

from (56). Then

$$(110) \quad [h_1, [h_2, \dots, [h_{k-1}, [h_k, h_{k+1}]] \dots]] = 0 \quad \forall h_1, \dots, h_{k+1} \in \ker \psi_k.$$

In particular, the kernel of ψ_k is nilpotent.

Proof. Let $n \geq 2q - 1$. Combining formulas (84), (86), (101) with the definition of $\varphi_n: \mathbb{F}^n(\mathcal{E}, a) \rightarrow \mathbb{F}^{n-1}(\mathcal{E}, a)$, we see that $\ker \varphi_n$ is generated by the elements \tilde{A}^l , $l \in \mathbb{Z}_{\geq 0}$. Then (109) follows from (105).

So we have proved that the kernel of the homomorphism $\varphi_n: \mathbb{F}^n(\mathcal{E}, a) \rightarrow \mathbb{F}^{n-1}(\mathcal{E}, a)$ is contained in the center of the Lie algebra $\mathbb{F}^n(\mathcal{E}, a)$ for any $n \geq 2q - 1$.

Let us prove (110) by induction on k . Since $\psi_1 = \varphi_{2q-1}$, for $k = 1$ property (110) follows from (109). Let $r \in \mathbb{Z}_{>0}$ be such that (110) is valid for $k = r$. Then for any $h'_1, h'_2, \dots, h'_{r+2} \in \ker \psi_{r+1}$ we have

$$(111) \quad [\varphi_{r+2q-1}(h'_2), [\varphi_{r+2q-1}(h'_3), \dots, [\varphi_{r+2q-1}(h'_r), [\varphi_{r+2q-1}(h'_{r+1}), \varphi_{r+2q-1}(h'_{r+2})]] \dots]] = 0,$$

because $\varphi_{r+2q-1}(h'_i) \in \ker \psi_r$ for $i = 2, 3, \dots, r+2$. Equation (111) says that

$$(112) \quad [h'_2, [h'_3, \dots, [h'_r, [h'_{r+1}, h'_{r+2}]] \dots]] \in \ker \varphi_{r+2q-1}.$$

Since $\ker \varphi_{r+2q-1}$ is contained in the center of $\mathbb{F}^{r+2q-1}(\mathcal{E}, a)$, property (112) yields

$$[h'_1, [h'_2, [h'_3, \dots, [h'_r, [h'_{r+1}, h'_{r+2}]] \dots]]] = 0.$$

So we have proved (110) for $k = r + 1$. Clearly, property (110) implies that $\ker \psi_k$ is nilpotent. \square

5. RELATIONS BETWEEN $\mathbb{F}^0(\mathcal{E}, a)$ AND THE WAHLQUIST-ESTABROOK PROLONGATION ALGEBRA

Consider an evolution equation of the form

$$(113) \quad u_t = F(u_0, u_1, \dots, u_d), \quad u = u(x, t), \quad u_k = \frac{\partial^k u}{\partial x^k}, \quad u_0 = u.$$

Note that the function F in (113) does not depend on x, t .

Let \mathcal{E} be the infinite prolongation of (113). Let

$$(114) \quad a = (x = x_0, t = t_0, u_k = a_k) \in \mathcal{E}, \quad x_0, t_0, a_k \in \mathbb{K}, \quad k \in \mathbb{Z}_{\geq 0},$$

be a point of \mathcal{E} .

The *Wahlquist-Estabrook prolongation algebra* of equation (113) at the point (114) can be defined in terms of generators and relations as follows. Consider formal power series

$$(115) \quad \mathcal{A} = \sum_{i \geq 0} (u_0 - a_0)^i \cdot \mathcal{A}_i, \quad \mathcal{B} = \sum_{j_0, \dots, j_{d-1} \geq 0} (u_0 - a_0)^{j_0} \dots (u_{d-1} - a_{d-1})^{j_{d-1}} \cdot \mathcal{B}_{j_0 \dots j_{d-1}},$$

where

$$(116) \quad \mathcal{A}_i, \quad \mathcal{B}_{j_0 \dots j_{d-1}}, \quad i, j_0, \dots, j_{d-1} \in \mathbb{Z}_{\geq 0},$$

are elements of a Lie algebra. The equation

$$(117) \quad D_x(\mathcal{B}) - D_t(\mathcal{A}) + [\mathcal{A}, \mathcal{B}] = 0$$

is equivalent to some Lie algebraic relations for (116). The Wahlquist-Estabrook prolongation algebra (WE algebra for short) is given by the generators (116) and these relations. Denote this algebra by \mathfrak{W} .

We are going to show that the algebra $\mathbb{F}^0(\mathcal{E}, a)$ for equation (113) is isomorphic to some subalgebra of \mathfrak{W} .

According to Remark 13, the algebra $\mathbb{F}^0(\mathcal{E}, a)$ is generated by $\mathbb{A}_i^{l_1, l_2}$, $\mathbb{B}_{j_0 \dots j_{d-1}}^{l_1, l_2}$ for $l_1, l_2, i, j_0, \dots, j_{d-1} \in \mathbb{Z}_{\geq 0}$. According to (53), one has $\mathbb{A}_0^{l_1, l_2} = \mathbb{B}_{0 \dots 0}^{l_1, l_2} = 0$ for all l_1, l_2 .

Since equation (113) is invariant with respect to the change of variables $x \mapsto x - x_0$, $t \mapsto t - t_0$, we can assume $x_0 = t_0 = 0$ in (114). Since $\mathbb{A}_0^{l_1, l_2} = \mathbb{B}_{0 \dots 0}^{0, l_2} = 0$ and $x_0 = t_0 = 0$, in the case $n = 0$ the power series (50), (51), (52) are written as

$$(118) \quad \mathbb{A} = \sum_{l_1, l_2 \geq 0, i > 0} x^{l_1} t^{l_2} (u_0 - a_0)^i \cdot \mathbb{A}_i^{l_1, l_2},$$

$$(119) \quad \mathbb{B} = \sum_{l_1, l_2, j_0, \dots, j_{d-1} \geq 0} x^{l_1} t^{l_2} (u_0 - a_0)^{j_0} \dots (u_{d-1} - a_{d-1})^{j_{d-1}} \cdot \mathbb{B}_{j_0 \dots j_{d-1}}^{l_1, l_2}, \quad \mathbb{B}_{0 \dots 0}^{0, l_2} = 0,$$

$$(120) \quad D_x(\mathbb{B}) - D_t(\mathbb{A}) + [\mathbb{A}, \mathbb{B}] = 0, \quad \mathbb{A}_i^{l_1, l_2}, \mathbb{B}_{j_0 \dots j_{d-1}}^{l_1, l_2} \in \mathbb{F}^0(\mathcal{E}, a).$$

The next lemma follows from the definition of $\mathbb{F}^0(\mathcal{E}, a)$.

Lemma 6. *Let \mathfrak{L} be a Lie algebra. Consider formal power series of the form*

$$P = \sum_{l_1, l_2 \geq 0, i > 0} x^{l_1} t^{l_2} (u_0 - a_0)^i \cdot P_i^{l_1, l_2},$$

$$Q = \sum_{l_1, l_2, j_0, \dots, j_{d-1} \geq 0} x^{l_1} t^{l_2} (u_0 - a_0)^{j_0} \dots (u_{d-1} - a_{d-1})^{j_{d-1}} \cdot Q_{j_0 \dots j_{d-1}}^{l_1, l_2},$$

$$P_i^{l_1, l_2}, Q_{j_0 \dots j_{d-1}}^{l_1, l_2} \in \mathfrak{L}, \quad Q_{0 \dots 0}^{0, l_2} = 0.$$

If $D_x(Q) - D_t(P) + [P, Q] = 0$, then the map $\mathbb{A}_i^{l_1, l_2} \mapsto P_i^{l_1, l_2}$, $\mathbb{B}_{j_0 \dots j_{d-1}}^{l_1, l_2} \mapsto Q_{j_0 \dots j_{d-1}}^{l_1, l_2}$ determines a homomorphism from $\mathbb{F}^0(\mathcal{E}, a)$ to \mathfrak{L} .

Let \mathfrak{L} be a Lie algebra. A formal ZCR of Wahlquist-Estabrook type with coefficients in \mathfrak{L} is given by formal power series

$$(121) \quad M = \sum_{i \geq 0} (u_0 - a_0)^i \cdot M_i, \quad N = \sum_{j_0, \dots, j_{d-1} \geq 0} (u_0 - a_0)^{j_0} \dots (u_{d-1} - a_{d-1})^{j_{d-1}} \cdot N_{j_0 \dots j_{d-1}},$$

$$M_i, N_{j_0 \dots j_{d-1}} \in \mathfrak{L},$$

satisfying

$$(122) \quad D_x(N) - D_t(M) + [M, N] = 0.$$

The next lemma follows from the definition of the WE algebra \mathfrak{W} .

Lemma 7. *Any formal ZCR of Wahlquist-Estabrook type (121), (122) with coefficients in \mathfrak{L} determines a homomorphism $\mathfrak{W} \rightarrow \mathfrak{L}$ given by $\mathcal{A}_i \mapsto M_i$, $\mathcal{B}_{j_0 \dots j_{d-1}} \mapsto N_{j_0 \dots j_{d-1}}$.*

Remark 14. For any Lie algebra \mathfrak{L} , there is a (possibly infinite-dimensional) vector space V such that \mathfrak{L} is isomorphic to a Lie subalgebra of $\mathfrak{gl}(V)$. Here $\mathfrak{gl}(V)$ is the algebra of linear maps $V \rightarrow V$.

For example, one can use the following construction. Denote by $U(\mathfrak{L})$ the universal enveloping algebra of \mathfrak{L} . We have the injective homomorphism of Lie algebras

$$\xi: \mathfrak{L} \hookrightarrow \mathfrak{gl}(U(\mathfrak{L})), \quad \xi(v)(w) = vw, \quad v \in \mathfrak{L}, \quad w \in U(\mathfrak{L}).$$

So one can set $V = U(\mathfrak{L})$.

Denote by \mathbf{F} the vector space of formal power series in variables z_1, z_2 with coefficients in $\mathbb{F}^0(\mathcal{E}, a)$. That is, an element of \mathbf{F} is a power series of the form

$$\sum_{l_1, l_2 \in \mathbb{Z}_{\geq 0}} z_1^{l_1} z_2^{l_2} C^{l_1 l_2}, \quad C^{l_1 l_2} \in \mathbb{F}^0(\mathcal{E}, a).$$

The space \mathbf{F} has the Lie algebra structure given by

$$\left[\sum_{l_1, l_2} z_1^{l_1} z_2^{l_2} C^{l_1 l_2}, \sum_{\tilde{l}_1, \tilde{l}_2} z_1^{\tilde{l}_1} z_2^{\tilde{l}_2} \tilde{C}^{\tilde{l}_1 \tilde{l}_2} \right] = \sum_{l_1, l_2, \tilde{l}_1, \tilde{l}_2} z_1^{l_1 + \tilde{l}_1} z_2^{l_2 + \tilde{l}_2} \left[C^{l_1 l_2}, \tilde{C}^{\tilde{l}_1 \tilde{l}_2} \right], \quad C^{l_1 l_2}, \tilde{C}^{\tilde{l}_1 \tilde{l}_2} \in \mathbb{F}^0(\mathcal{E}, a).$$

We have also the following homomorphism of Lie algebras

$$(123) \quad \nu: \mathbf{F} \rightarrow \mathbb{F}^0(\mathcal{E}, a), \quad \nu\left(\sum_{l_1, l_2 \in \mathbb{Z}_{\geq 0}} z_1^{l_1} z_2^{l_2} C^{l_1 l_2}\right) = C^{00}.$$

For $i = 1, 2$, let $\partial_{z_i}: \mathbf{F} \rightarrow \mathbf{F}$ be the linear map given by

$$\partial_{z_i}\left(\sum z_1^{l_1} z_2^{l_2} C^{l_1 l_2}\right) = \sum \frac{\partial}{\partial z_i}\left(z_1^{l_1} z_2^{l_2}\right) C^{l_1 l_2}.$$

Let \mathbb{D} be the linear span of $\partial_{z_1}, \partial_{z_2}$ in the vector space of linear maps $\mathbf{F} \rightarrow \mathbf{F}$. Since the maps $\partial_{z_1}, \partial_{z_2}$ commute, the space \mathbb{D} is a 2-dimensional abelian Lie algebra with respect to the commutator of maps.

Denote by \mathbb{L} the vector space $\mathbb{D} \oplus \mathbf{F}$ with the following Lie algebra structure

$$[X_1 + f_1, X_2 + f_2] = X_1(f_2) - X_2(f_1) + [f_1, f_2], \quad X_1, X_2 \in \mathbb{D}, \quad f_1, f_2 \in \mathbf{F}.$$

An element of \mathbb{L} can be written as a sum of the following form

$$(y_1 \partial_{z_1} + y_2 \partial_{z_2}) + \sum z_1^{l_1} z_2^{l_2} C^{l_1 l_2}, \quad y_1, y_2 \in \mathbb{K}, \quad C^{l_1 l_2} \in \mathbb{F}^0(\mathcal{E}, a).$$

Theorem 9. *Let $\mathfrak{R} \subset \mathfrak{W}$ be the subalgebra generated by the elements*

$$(124) \quad (\text{ad } \mathcal{A}_0)^k(\mathcal{A}_i), \quad k \in \mathbb{Z}_{\geq 0}, \quad i \in \mathbb{Z}_{>0}.$$

Then the map $(\text{ad } \mathcal{A}_0)^k(\mathcal{A}_i) \mapsto k! \cdot \mathbb{A}_i^{k,0}$ determines an isomorphism between \mathfrak{R} and $\mathbb{F}^0(\mathcal{E}, a)$.

Proof. Recall that $D_x = \frac{\partial}{\partial x} + \sum_{k \geq 0} u_{k+1} \frac{\partial}{\partial u_k}$ and $D_t = \frac{\partial}{\partial t} + \sum_{k \geq 0} D_x^k(F) \frac{\partial}{\partial u_k}$. Equation (120) is equivalent to

$$(125) \quad \sum_{l_1, l_2, j_0, \dots, j_{d-1}} \frac{\partial}{\partial x}(x^{l_1} t^{l_2})(u_0 - a_0)^{j_0} \dots (u_{d-1} - a_{d-1})^{j_{d-1}} \cdot \mathbb{B}_{j_0 \dots j_{d-1}}^{l_1, l_2} +$$

$$+ \sum_{l_1, l_2, j_0, \dots, j_{d-1}} x^{l_1} t^{l_2} D_x((u_0 - a_0)^{j_0} \dots (u_{d-1} - a_{d-1})^{j_{d-1}}) \cdot \mathbb{B}_{j_0 \dots j_{d-1}}^{l_1, l_2}$$

$$- \sum_{l_1, l_2, i} \frac{\partial}{\partial t}(x^{l_1} t^{l_2})(u_0 - a_0)^i \cdot \mathbb{A}_i^{l_1, l_2} - \sum_{l_1, l_2, i} x^{l_1} t^{l_2} D_t((u_0 - a_0)^i) \cdot \mathbb{A}_i^{l_1, l_2} + [\mathbb{A}, \mathbb{B}] = 0.$$

We regard the expressions

$$(126) \quad \tilde{\mathcal{A}} = \partial_{z_1} + \sum_{i > 0} (u_0 - a_0)^i \cdot \left(\sum_{l_1, l_2} z_1^{l_1} z_2^{l_2} \mathbb{A}_i^{l_1, l_2} \right),$$

$$(127) \quad \tilde{\mathcal{B}} = \left(\partial_{z_2} + \sum_{l_1, l_2} z_1^{l_1} z_2^{l_2} \mathbb{B}_{0 \dots 0}^{l_1, l_2} \right) + \sum_{\substack{j_0, \dots, j_{d-1} \geq 0, \\ j_0 + \dots + j_{d-1} > 0}} (u_0 - a_0)^{j_0} \dots (u_{d-1} - a_{d-1})^{j_{d-1}} \cdot \left(\sum_{l_1, l_2} z_1^{l_1} z_2^{l_2} \mathbb{B}_{j_0 \dots j_{d-1}}^{l_1, l_2} \right)$$

as formal power series with coefficients in \mathbb{L} .

Since the function F in (113) does not depend on x and t , equation (125) is equivalent to $D_x(\tilde{\mathcal{B}}) - D_t(\tilde{\mathcal{A}}) + [\tilde{\mathcal{A}}, \tilde{\mathcal{B}}] = 0$, which implies that the power series (126), (127) constitute a formal ZCR of Wahlquist-Estabrook type with coefficients in \mathbb{L} .

Applying Lemma 7 to this formal ZCR, we obtain the homomorphism

$$(128) \quad \varphi: \mathfrak{W} \rightarrow \mathbb{L}, \quad \varphi(\mathcal{A}_0) = \partial_{z_1}, \quad \varphi(\mathcal{A}_i) = \sum_{l_1, l_2} z_1^{l_1} z_2^{l_2} \mathbb{A}_i^{l_1, l_2}, \quad i > 0,$$

$$\varphi(\mathcal{B}_{0 \dots 0}) = \left(\partial_{z_2} + \sum_{l_1, l_2} z_1^{l_1} z_2^{l_2} \mathbb{B}_{0 \dots 0}^{l_1, l_2} \right), \quad \varphi(\mathcal{B}_{j_0 \dots j_{d-1}}) = \left(\sum_{l_1, l_2} z_1^{l_1} z_2^{l_2} \mathbb{B}_{j_0 \dots j_{d-1}}^{l_1, l_2} \right), \quad j_0 + \dots + j_{d-1} > 0.$$

Clearly, \mathbf{F} is a Lie subalgebra of $\mathbb{L} = \mathbf{D} \oplus \mathbf{F}$. In view of (128), for any $k \in \mathbb{Z}_{\geq 0}$ and $i \in \mathbb{Z}_{>0}$ we have

$$(129) \quad \varphi\left((\text{ad } \mathcal{A}_0)^k(\mathcal{A}_i)\right) = (\text{ad } \partial_{z_1})^k \left(\sum_{l_1, l_2} z_1^{l_1} z_2^{l_2} \mathbb{A}_i^{l_1, l_2} \right) = (\partial_{z_1})^k \left(\sum_{l_1, l_2} z_1^{l_1} z_2^{l_2} \mathbb{A}_i^{l_1, l_2} \right) \in \mathbf{F}.$$

Since $\mathfrak{R} \subset \mathfrak{W}$ is generated by the elements (124), property (129) implies $\varphi(\mathfrak{R}) \subset \mathbf{F} \subset \mathbb{L}$. Using the homomorphism (123) and property (129), we obtain

$$(130) \quad \nu \circ \varphi|_{\mathfrak{R}}: \mathfrak{R} \rightarrow \mathbb{F}^0(\mathcal{E}, a), \quad (\nu \circ \varphi)\left((\text{ad } \mathcal{A}_0)^k(\mathcal{A}_i)\right) = k! \cdot \mathbb{A}_i^{k, 0}, \quad k \in \mathbb{Z}_{\geq 0}, \quad i \in \mathbb{Z}_{>0}.$$

Using Remark 14, we can assume that \mathfrak{W} is embedded in the algebra $\mathfrak{gl}(V)$ for some vector space V . Let \mathbf{S} be the vector space of power series of the form

$$(131) \quad \sum_{l_1, l_2, i_0, \dots, i_k \geq 0} x^{l_1} t^{l_2} (u_0 - a_0)^{i_0} \dots (u_k - a_k)^{i_k} \cdot C_{i_0 \dots i_k}^{l_1, l_2}, \quad C_{i_0 \dots i_k}^{l_1, l_2} \in \mathfrak{gl}(V), \quad k \in \mathbb{Z}_{\geq 0}.$$

Note that \mathbf{S} contains power series (131) for all $k \in \mathbb{Z}_{\geq 0}$. For each $C \in \mathbf{S}$, the power series $D_x(C), D_t(C) \in \mathbf{S}$ are defined according to Remark 11.

Recall that $\mathfrak{gl}(V)$ consists of linear maps $V \rightarrow V$. Since $\mathfrak{gl}(V)$ is an associative algebra with respect to the composition of maps, the space \mathbf{S} is an associative algebra with respect to the standard multiplication of formal power series.

Also, using Remark 11 and the Lie bracket on $\mathfrak{gl}(V)$, we obtain a Lie bracket on the space \mathbf{S} .

Set $\mathcal{B}_0 = \mathcal{B}_{0 \dots 0}$, where $\mathcal{B}_{0 \dots 0}$ is the free term of the power series \mathcal{B} from (115). Since $\mathcal{A}_i, \mathcal{B}_{j_0 \dots j_{d-1}} \in \mathfrak{W} \subset \mathfrak{gl}(V)$ for all $i, j_0, \dots, j_{d-1} \in \mathbb{Z}_{\geq 0}$, the power series $e^{x\mathcal{A}_0}, e^{t\mathcal{B}_0}$, and (115) belong to \mathbf{S} . Set

$$(132) \quad P = -e^{t\mathcal{B}_0} \mathcal{A}_0 e^{-t\mathcal{B}_0} + e^{t\mathcal{B}_0} e^{x\mathcal{A}_0} \mathcal{A} e^{-x\mathcal{A}_0} e^{-t\mathcal{B}_0},$$

$$(133) \quad Q = -\mathcal{B}_0 + e^{t\mathcal{B}_0} e^{x\mathcal{A}_0} \mathcal{B} e^{-x\mathcal{A}_0} e^{-t\mathcal{B}_0}.$$

Using (132), (133), we get

$$(134) \quad D_x(Q) = e^{t\mathcal{B}_0} [\mathcal{A}_0, e^{x\mathcal{A}_0} \mathcal{B} e^{-x\mathcal{A}_0}] e^{-t\mathcal{B}_0} + e^{t\mathcal{B}_0} e^{x\mathcal{A}_0} D_x(\mathcal{B}) e^{-x\mathcal{A}_0} e^{-t\mathcal{B}_0},$$

$$(135) \quad D_t(P) = -[\mathcal{B}_0, e^{t\mathcal{B}_0} \mathcal{A}_0 e^{-t\mathcal{B}_0}] + [\mathcal{B}_0, e^{t\mathcal{B}_0} e^{x\mathcal{A}_0} \mathcal{A} e^{-x\mathcal{A}_0} e^{-t\mathcal{B}_0}] + e^{t\mathcal{B}_0} e^{x\mathcal{A}_0} D_t(\mathcal{A}) e^{-x\mathcal{A}_0} e^{-t\mathcal{B}_0}.$$

Recall that $D_x(\mathcal{B}) - D_t(\mathcal{A}) + [\mathcal{A}, \mathcal{B}] = 0$ according to (117). Combining this with (132), (133), (134), (135), one obtains

$$(136) \quad D_x(Q) - D_t(P) + [P, Q] = e^{t\mathcal{B}_0} e^{x\mathcal{A}_0} (D_x(\mathcal{B}) - D_t(\mathcal{A}) + [\mathcal{A}, \mathcal{B}]) e^{-x\mathcal{A}_0} e^{-t\mathcal{B}_0} = 0.$$

Formulas (115), (132), (133) yield

$$(137) \quad P = -e^{t\mathcal{B}_0} \mathcal{A}_0 e^{-t\mathcal{B}_0} + \sum_{i \geq 0} (u_0 - a_0)^i \cdot e^{t\mathcal{B}_0} e^{x\mathcal{A}_0} \mathcal{A}_i e^{-x\mathcal{A}_0} e^{-t\mathcal{B}_0} =$$

$$= \sum_{l_1, l_2 \geq 0, i > 0} x^{l_1} t^{l_2} (u_0 - a_0)^i \frac{1}{l_1! l_2!} (\text{ad } \mathcal{B}_0)^{l_2} \left((\text{ad } \mathcal{A}_0)^{l_1} (\mathcal{A}_i) \right).$$

$$(138) \quad Q = -\mathcal{B}_0 + \sum_{j_0, \dots, j_{d-1} \geq 0} (u_0 - a_0)^{j_0} \dots (u_{d-1} - a_{d-1})^{j_{d-1}} \cdot e^{t\mathcal{B}_0} e^{x\mathcal{A}_0} \mathcal{B}_{j_0 \dots j_{d-1}} e^{-x\mathcal{A}_0} e^{-t\mathcal{B}_0} =$$

$$= -\mathcal{B}_0 + \sum_{l_1, l_2, j_0, \dots, j_{d-1} \geq 0} x^{l_1} t^{l_2} (u_0 - a_0)^{j_0} \dots (u_{d-1} - a_{d-1})^{j_{d-1}} \cdot \frac{1}{l_1! l_2!} (\text{ad } \mathcal{B}_0)^{l_2} \left((\text{ad } \mathcal{A}_0)^{l_1} (\mathcal{B}_{j_0 \dots j_{d-1}}) \right).$$

From (136), (137), (138) it follows that the power series P, Q satisfy all conditions of Lemma 6. Applying Lemma 6 to P, Q given by (137), (138), we obtain the homomorphism

(139)

$$\begin{aligned} \psi: \mathbb{F}^0(\mathcal{E}, a) &\rightarrow \mathfrak{W}, & \psi(\mathbb{A}_i^{l_1, l_2}) &= \frac{1}{l_1! l_2!} (\text{ad } \mathcal{B}_0)^{l_2} \left((\text{ad } \mathcal{A}_0)^{l_1} (\mathcal{A}_i) \right), & l_1, l_2 &\in \mathbb{Z}_{\geq 0}, \quad i \in \mathbb{Z}_{>0}, \\ \psi(\mathbb{B}_{j_0 \dots j_{d-1}}^{l_1, l_2}) &= \frac{1}{l_1! l_2!} (\text{ad } \mathcal{B}_0)^{l_2} \left((\text{ad } \mathcal{A}_0)^{l_1} (\mathcal{B}_{j_0 \dots j_{d-1}}) \right), & l_1, l_2, j_0, \dots, j_{d-1} &\in \mathbb{Z}_{\geq 0}, \quad j_0 + \dots + j_{d-1} > 0, \\ \psi(\mathbb{B}_{0 \dots 0}^{l'_1, l'_2}) &= \frac{1}{l'_1! l'_2!} (\text{ad } \mathcal{B}_0)^{l'_2} \left((\text{ad } \mathcal{A}_0)^{l'_1} (\mathcal{B}_{0 \dots 0}) \right), & l'_1 &\in \mathbb{Z}_{>0}, \quad l'_2 \in \mathbb{Z}_{\geq 0}. \end{aligned}$$

From (139) we get

$$(140) \quad \psi(\mathbb{A}_i^{l_1, 0}) = \frac{1}{l_1!} (\text{ad } \mathcal{A}_0)^{l_1} (\mathcal{A}_i) \in \mathfrak{R}, \quad l_1 \in \mathbb{Z}_{\geq 0}, \quad i \in \mathbb{Z}_{>0}.$$

Since, by Proposition 5, the elements $\mathbb{A}_i^{l_1, 0}$, $l_1 \in \mathbb{Z}_{\geq 0}$, $i \in \mathbb{Z}_{>0}$, generate the algebra $\mathbb{F}^0(\mathcal{E}, a)$, property (140) implies $\psi(\mathbb{F}^0(\mathcal{E}, a)) \subset \mathfrak{R}$. Then from (130), (140) it follows that the homomorphisms $\psi: \mathbb{F}^0(\mathcal{E}, a) \rightarrow \mathfrak{R}$ and $\nu \circ \varphi|_{\mathfrak{R}}: \mathfrak{R} \rightarrow \mathbb{F}^0(\mathcal{E}, a)$ are inverse to each other. \square

6. THE ALGEBRAS $\mathbb{F}^n(\mathcal{E}, a)$ FOR THE KdV EQUATION

Consider the infinite-dimensional Lie algebra

$$\mathfrak{sl}_2(\mathbb{K}[\lambda]) \cong \mathfrak{sl}_2(\mathbb{K}) \otimes_{\mathbb{K}} \mathbb{K}[\lambda],$$

where $\mathbb{K}[\lambda]$ is the algebra of polynomials in λ .

Theorem 10. *Let \mathcal{E} be the infinite prolongation of the KdV equation $u_t = u_{xxx} + u_x u$. Let $a \in \mathcal{E}$. Then $\mathbb{F}^0(\mathcal{E}, a)$ is isomorphic to the direct sum of $\mathfrak{sl}_2(\mathbb{K}[\lambda])$ and a 3-dimensional abelian Lie algebra.*

For each $n \in \mathbb{Z}_{>0}$, consider the homomorphism $\varphi_n: \mathbb{F}^n(\mathcal{E}, a) \rightarrow \mathbb{F}^{n-1}(\mathcal{E}, a)$ from (56) and the homomorphism $\psi_n: \mathbb{F}^n(\mathcal{E}, a) \rightarrow \mathbb{F}^0(\mathcal{E}, a)$ that is equal to the composition of

$$\mathbb{F}^n(\mathcal{E}, a) \rightarrow \mathbb{F}^{n-1}(\mathcal{E}, a) \rightarrow \dots \rightarrow \mathbb{F}^1(\mathcal{E}, a) \rightarrow \mathbb{F}^0(\mathcal{E}, a).$$

Then the kernel of φ_n is contained in the center of the Lie algebra $\mathbb{F}^n(\mathcal{E}, a)$, and the kernel of ψ_n is nilpotent.

Proof. Let \mathfrak{W} be the Wahlquist-Estabrook prolongation algebra of the KdV equation. According to [2, 3], the algebra \mathfrak{W} is isomorphic to the direct sum of $\mathfrak{sl}_2(\mathbb{K}[\lambda])$ and a 5-dimensional nilpotent Lie algebra.

Consider the subalgebra $\mathfrak{R} \subset \mathfrak{W}$ defined in Theorem 9. According to Theorem 9, one has $\mathbb{F}^0(\mathcal{E}, a) \cong \mathfrak{R}$. From the description of \mathfrak{W} in [2, 3] it follows that \mathfrak{R} is isomorphic to the direct sum of $\mathfrak{sl}_2(\mathbb{K}[\lambda])$ and a 3-dimensional abelian Lie algebra.

The results about the homomorphisms $\varphi_n: \mathbb{F}^n(\mathcal{E}, a) \rightarrow \mathbb{F}^{n-1}(\mathcal{E}, a)$ and $\psi_n: \mathbb{F}^n(\mathcal{E}, a) \rightarrow \mathbb{F}^0(\mathcal{E}, a)$ follow from Theorem 8 in the case $q = 1$. \square

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